An efficient numerical algorithm for the 3D wave equation in domains of complex shape

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Abstract

We propose an efficient finite difference algorithm for the 3D wave equation in domains with curvilinear boundaries. Our approach combines the method of difference potentials for handling the complex geometries on regular grids and the Huygens' principle for time marching.

Keywords: method of difference potentials, Huygens' principle, wave equation

1 Introduction

We consider an initial boundary value problem (IBVP) for the 3D wave equation:

$$u_{tt} = c^2 \Delta u, \quad (\boldsymbol{x}, t) \in \Omega \times [0, T], \tag{1a}$$

$$\text{B.C. on } \partial\Omega \times [0, T], \quad \text{I.C. at } t = 0, \tag{1b}$$

where c is the speed of light and Δ is the Laplacian. The boundary conditions set on the walls of the curvilinear cylinder $\Gamma = \partial\Omega \times [0, T]$ may depend on time. The computational domain Ω may have a complex shape in 3D in the sense that its boundary $\partial\Omega$ does not have conform to the (regular) discretization grid, see Fig. 1.

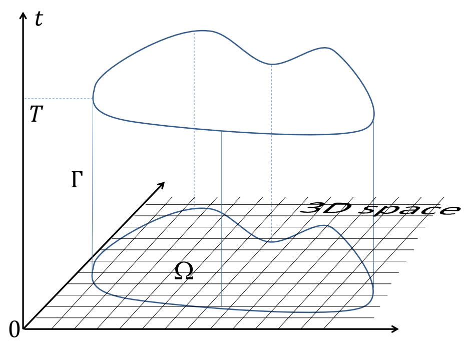

Figure 1: Computational domain (schematic).

The approaches to solving IBVPs of type (1) include various discretizations over Ω (e.g., finite volumes, DG) that have to exercise a case-by-case care for the geometry and specific boundary conditions (BCs) in (1b), as well as the time-dependent BEM that becomes progressively more costly as the time elapses and is also sensitive to the type of boundary conditions.

We propose an easy to implement finite difference time domain algorithm capable of handling complex non-conforming boundaries and arbitrary boundary conditions on regular grids with no loss of accuracy. Moreover, for the governing PDEs that admit the diffusionless propagation of waves (i.e., satisfy the Huygens' principle), *the proposed algorithm has a provably better asymptotic complexity in long runs than even the plain explicit time marching over* Ω *regardless of the type of discretization* (finite differences, finite volumes, FEM, DG). The reason is that the original 3D problem is efficiently reduced from the domain to its boundary only.

Our approach employs the method of difference potentials (MDP) [1] that has previously been used for steady-state problems, e.g., the Helmholtz equation [2]. The novel contribution of this paper is the time marching algorithm that is particularly efficient for Huygens' PDEs as it exploits the lacunae in their solutions. In our prior work, we have used lacunae for handling the artificial outer boundaries [3, 4].

2 Method

The MDP equivalently reduces the PDE (1a) from its domain $\Omega \times [0, T]$ to the operator equation at the boundary $\Gamma = \partial\Omega \times [0, T]$:

$$\boldsymbol{P}_\Gamma \boldsymbol{\xi}_\Gamma = \boldsymbol{\xi}_\Gamma, \tag{2}$$

where $\boldsymbol{P}_\Gamma$ is a Calderon's projection and $\boldsymbol{\xi}_\Gamma \equiv (\xi_0, \xi_1)$ is the density of a generalized Calderon's potential. The functions ξ_0 and ξ_1 can be interpreted as traces of the solution u and its normal derivative on Γ, respectively. The boundary equation (2), which is equivalent to (1a), is solved as a system along with the BC from (1b), which can be arbitrary as long as the overall formulation (1) is well-posed. In simple cases, the BC explicitly provides some component of $\boldsymbol{\xi}_\Gamma$, e.g., ξ_0 in the case of a Dirichlet BC and ξ_1 in the case of a Neumann BC. The remaining component is then obtained as a solution to (2).

The MDP enables an efficient solution of the boundary equation (2). It also allows one to easily restore the solution u on the entire Ω at T_{final} using $\boldsymbol{\xi}_\Gamma$. The solution of (2) requires solving a number of inhomogeneous auxiliary problems (APs) for equation (1a) formulated on a larger domain Ω_0 that has simple shape, see Fig. 2. The boundary $\partial\Omega_0$ conforms to the grid and as such, the APs can be easily integrated by any appropriate finite difference scheme.

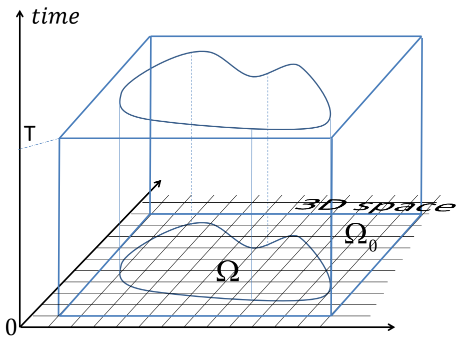

Figure 2: Computational domain for the AP.

The key component of our time marching algorithm is the use of the Huygens' principle, which implies that for a finite domain in space, the extent of the domain of dependence of equation (1a) in time is also finite. This property allows one to solve (2) over long computational times $T_{\text{final}} \gg T$ sequentially, updating the density $\boldsymbol{\xi}_\Gamma$ by "chunks" of size T, see Fig. 3.

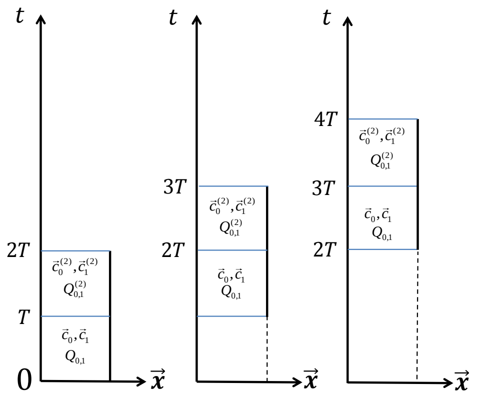

Figure 3: Time marching by "chunks" of size T.

In doing so, the time marching is done only along the boundary Γ, which effectively reduces the space dimension of the evolution scheme for $\boldsymbol{\xi}_\Gamma$ by one compared to the conventional time marching of the solution over the entire 3D domain Ω. The solution u on Ω is computed only once, at the final moment $t = T_{\text{final}}$. Due to the reduced dimension and the special choice of an economical basis on Γ, the proposed method appears more efficient in long runs than the standard explicit time marching over Ω.

3 Numerical simulations

We have tested the proposed method for a variety of IBVPs (1) where the domain Ω was a ball while the discretization grid was Cartesian. In all the cases, we have obtained stable performance over long times and the design rate of grid convergence that corresponds to that of the core scheme used in MDP (second order for our current simulations). In Fig. 4, we are showing the convergence for a Robin BC in (1b).

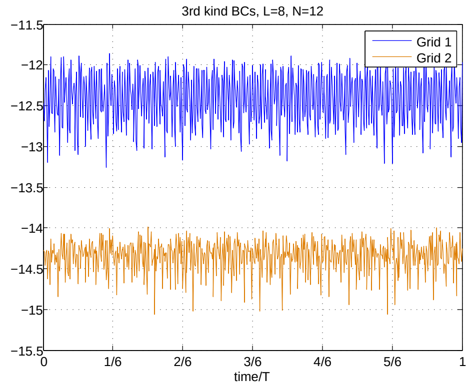

Figure 4: Grid convergence for a Robin BC.

4 Future work

In the future, we will consider exterior problems and high order accurate schemes.

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