Lacunae-based computation of time-harmonic scattering in 3D

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Abstract

We propose a method for the numerical solution of 3D time-harmonic scattering problems which (i) handles complex scattering shapes using high-order finite differences on regular Cartesian grids; (ii) guarantees a perfect non-reflecting behavior at the artificial outer boundary; (iii) has a reusable core that allows one to recompute the solution with new boundary conditions conveniently and at a low cost.

Keywords: Calderon's boundary equations with projections, method of difference potentials (MDP), non-conforming scattering shapes, CAD surface, lacunae of hyperbolic equations.

1 Introduction

We consider single-frequency scattering of a scalar field about an obstacle embedded in a homogeneous medium. The scattered field u in the exterior of the scatterer Ω satisfies the Helmholtz equation with a constant wavenumber k:

$$\Delta u + k^2 u = 0, \quad \boldsymbol{x} \in \tilde{\Omega} = \mathbb{R}^3 \setminus \Omega, \tag{1a}$$

$$\boldsymbol{l}_\Gamma u = \phi, \quad \boldsymbol{x} \in \Gamma \equiv \partial\Omega, \tag{1b}$$

$$\partial_{|\boldsymbol{x}|} u - iku = o(|\boldsymbol{x}|^{-1}), \quad \boldsymbol{x} \to \infty, \tag{1c}$$

subject to the boundary condition (1b) on the surface Γ of the scatterer Ω and the Sommerfeld radiation condition at infinity (1c).

2 Surface parametrization

The scattering surface is composed of a number of non-intersecting patches, each represented as a 2D spline (NURBS) using CAD software, see Figure 1.

3 Method of difference potentials (MDP)

Define the Calderon potential for equation (1a):

$$\boldsymbol{P}_{\tilde{\Omega}} \boldsymbol{\xi}_\Gamma(\boldsymbol{x}) = \int_\Gamma \{\xi_1(\boldsymbol{y}) G(\boldsymbol{x}-\boldsymbol{y}) - \xi_0(\boldsymbol{y}) \frac{\partial G}{\partial \boldsymbol{n}}(\boldsymbol{x}-\boldsymbol{y})\} dS_{\boldsymbol{y}}, \tag{2}$$

where the density $\boldsymbol{\xi}_\Gamma = (\xi_0, \xi_1)$ is a vector function on the boundary Γ. Using the MDP, problem (1) is reduced to the boundary equation with projection for the density $\boldsymbol{\xi}_\Gamma$:

$$\boldsymbol{P}_\Gamma \boldsymbol{\xi}_\Gamma = \boldsymbol{\xi}_\Gamma \tag{3a}$$

$$\boldsymbol{l}_\Gamma \boldsymbol{\xi}_\Gamma = \phi, \tag{3b}$$

where $\boldsymbol{P}_\Gamma \stackrel{\text{def}}{=} \boldsymbol{Tr}_\Gamma \boldsymbol{P}_{\tilde{\Omega}}$ is the Calderon projection defined as the vector trace of the potential (2), $\boldsymbol{Tr}_\Gamma w \stackrel{\text{def}}{=} \left(w, \frac{\partial w}{\partial \boldsymbol{n}}\right)\big|_\Gamma$, and equation (3b) is the boundary condition (1b) recast in terms of the density $\boldsymbol{\xi}_\Gamma$. Once the density has been determined, the solution to (1) is given by $u = \boldsymbol{P}_{\tilde{\Omega}} \boldsymbol{\xi}_\Gamma$.

It is important that, instead of evaluating the integral (2), one can compute $\boldsymbol{P}_{\tilde{\Omega}} \boldsymbol{\xi}_\Gamma$ and $\boldsymbol{P}_\Gamma \boldsymbol{\xi}_\Gamma$, see (3a), by solving the auxiliary problem (AP) formulated on the entire $\mathbb{R}^3$:

$$\Delta v + k^2 v = f, \quad \boldsymbol{x} \in \mathbb{R}^3, \tag{4a}$$

$$\partial_{|\boldsymbol{x}|} v - ikv = o(|\boldsymbol{x}|^{-1}), \quad \boldsymbol{x} \to \infty, \tag{4b}$$

where the RHS f of equation (4a) is defined with the help of a sufficiently smooth compactly supported function w such that $\boldsymbol{Tr}_\Gamma w = \boldsymbol{\xi}_\Gamma$. Namely, $f = (\Delta w + k^2 w)$ on $\tilde{\Omega}$ and $f = 0$ on Ω. Then, $\boldsymbol{P}_{\tilde{\Omega}} \boldsymbol{\xi}_\Gamma = w - v$.

In practice, both ξ_0 and ξ_1 are represented as a truncated expansion with respect to an appropriate basis, e.g., Fourier or Chebyshev, $\boldsymbol{\xi}_\Gamma = \sum_{s=1}^N c_{0,s} \boldsymbol{\psi}_{0,s} + c_{1,s} \boldsymbol{\psi}_{1,s}$, where $\boldsymbol{\psi}_{0,s} = (\psi_s, 0)$ and $\boldsymbol{\psi}_{1,s} = (0, \psi_s)$. Then, (3b) becomes a relation between the coefficients $\boldsymbol{c}_0 = (c_{0,1}, ..., c_{0,N})$ and $\boldsymbol{c}_1 = (c_{1,1}, ..., c_{1,N})$.

By linearity, solution to the AP (4) can be written as $v = \sum_{s=1}^N c_{0,s} v_{0,s} + c_{1,s} v_{1,s}$, where $v_{0,s}$, $v_{1,s}$ solve (4) with the RHSs that correspond to the individual basis functions $\boldsymbol{\psi}_{0,s}$, $\boldsymbol{\psi}_{1,s}$. Solutions to these $2N$ subproblems are computed independently from one another on a finite computational domain $\Omega^{\text{aux}} \supset \Omega$ of a simple shape (e.g., a cube) using finite differences on a regular Cartesian grid, as explained in Section 4.

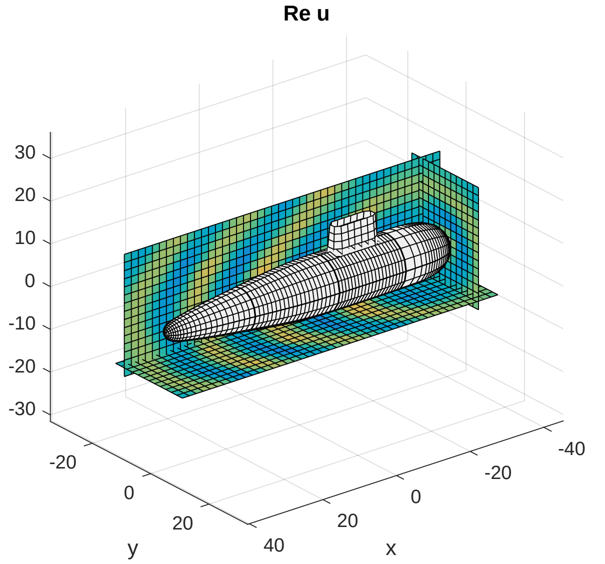

Figure 1: Plane wave sound-soft scattering about a submarine-like shape.

Then, equation (3a) can be recast as

$$\boldsymbol{Q}_0 \boldsymbol{c}_0 + \boldsymbol{Q}_1 \boldsymbol{c}_1 = 0, \tag{5}$$

where $\boldsymbol{Q}_0$ and $\boldsymbol{Q}_1$ are matrices with N columns given by the numerical solutions $v_{0,s}$ and $v_{1,s}$ sampled on the grid boundary, which is a specially chosen fringe of nodes of the discretization grid that straddles the continuous boundary Γ, see [1] for detail. Equation (5) along with the boundary condition (3b) is solved by least squares for the unknowns $\boldsymbol{c}_0$, $\boldsymbol{c}_1$. The matrices $\boldsymbol{Q}_0$ and $\boldsymbol{Q}_1$ are computed only once ahead of time. They represent a reusable core of the algorithm. It enables an inexpensive recalculation of the solution subject to a new boundary condition on the scatterer.

4 Lacunae-based solution of the AP

An outgoing solution to the wave equation:

$$\frac{1}{c^2}\frac{\partial^2 v}{\partial t^2} - \Delta v = -f(\boldsymbol{x})e^{-i\omega t}, \tag{6}$$

with a compactly supported in space RHS $-f(\boldsymbol{x})e^{-i\omega t}$ and subject to zero initial conditions, differs by factor $e^{-i\omega t}$ from the solution to the AP (4) inside the *secondary lacuna* of the wave equation, see [2, p. 467]. The lacuna is the space-time region behind the aft front of the outgoing wave. Hence, to compute v on the domain of interest Ω^{int}, $\Omega \subset \Omega^{\text{int}} \subset \Omega^{\text{aux}}$, it is sufficient to integrate the wave equation (6) over a finite time interval $\text{diam}(\Omega^{\text{int}})/c$, after which the domain of interest Ω^{int} completely falls into the lacuna. By placing the artificial outer boundary $\partial\Omega^{\text{aux}}$ far enough from Ω^{int}, one can make sure that no numerical reflection from $\partial\Omega^{\text{aux}}$ will reach Ω^{int} during the integration time regardless of boundary conditions on $\partial\Omega^{\text{aux}}$. The solution to the wave equation (6) computed this way is cleared from $e^{-i\omega t}$ at the last time step yielding the solution v to (4) on Ω^{int}.

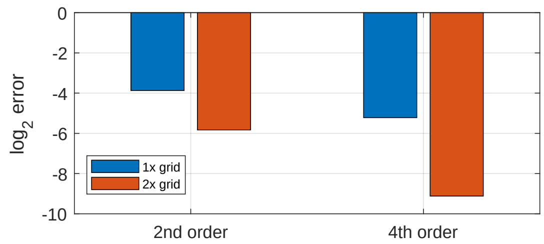

Figure 2: Grid convergence for the second- and fourth-order accurate finite-difference schemes. Boundary condition (1b) is of a Dirichlet type.

This approach to solving the Helmholtz equation, based on casting it into the time domain, resolves the long-standing issue of radiation boundary conditions (see, e.g., [3]) with perfect accuracy, guaranteeing no reflections at all from the outer boundary. As far as the general use of lacunae of hyperbolic equations for the numerical simulation of unsteady waves on unbounded regions, see [1] and the references therein.

5 Computational results

An example of a scattering solution is presented in Figure 1. Figure 2 corroborates the design rate of grid convergence for the central-difference second-order accurate scheme and compact fourth-order accurate scheme.

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