High order accurate solution of the wave equation by compact finite differences and difference potentials

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Abstract

High order numerical methods exhibit dramatic gains in efficiency over low order methods by providing better accuracy on coarse grids, and therefore the computation time needed to obtain a desired level of accuracy in simulations is greatly reduced. In addition to the increased convergence rate, it has been shown that high order methods result in smaller dispersion errors than low order methods. In order to fit the needs of physical problems, high order methods must exhibit several capabilities, such as handling variable coefficient operators, realistic geometries, and different types of boundary conditions. We demonstrate a flexible approach that efficiently solves second order hyperbolic PDEs with high order accuracy through the combined methodology of compact high order finite differences and difference potentials.

Keywords: high order accuracy, non-conforming boundaries, time-dependent waves, variable wave speed

1 Introduction

Consider the wave equation

$$u_{tt} = c^2\Delta u + F(x, y, t), \qquad (1)$$

where F is an inhomogeneous term and the wave-speed c may vary in space but not in time. Time discretization by the θ-method with $\theta = \frac{1}{12}$ yields a temporally fourth order implicit scheme. At each time step, one must solve an elliptic spatial equation in the form of the modified Helmholtz equation,

$$\Delta u - Ku = G, \qquad (2)$$

where G depends on the inhomogeneous term F as well as the solution at two previous time steps, and $K = \frac{1}{\theta c^2 h_t^2}$ where h_t is the time step. When $\theta = \frac{1}{12}$, the scheme is conditionally stable and fourth order accurate in time, while choosing $\theta \geq \frac{1}{4}$ yields an unconditionally stable scheme which is only second order in time. At each time step, equation (2) can be interpreted as a steady-state equation. We propose to solve it by compact high order finite differences and the method of difference potentials [4]. This is a distinctly different approach than that of [3], where the method of difference potentials is applied directly to the unsteady wave equation in 3+1 dimensional space-time.

2 Compact finite differences

Finite difference schemes on regular structured grids are a straightforward and efficient way to achieve high order accuracy for variable coefficient equations such as (2). Compact schemes enable high order accuracy without increasing the stencil size, and this simplifies the treatment of boundary conditions since the stencil will not extend beyond the boundary at the near-boundary nodes, see Figure 1. Compact

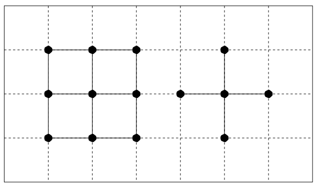

Figure 1: 2D compact (left) and five-point (right) stencils.

schemes also yield matrices with lower bandwidths than those resulting from wider stencils, and this reduced bandwidth improves the efficiency of solving the resulting linear system. A major limitation of conventional or compact finite differences is that they lose accuracy on domains which do not coincide with the discretization grid, and we address this by the method of difference potentials [4].

A compact 4th order Cartesian scheme for the Helmholtz equation (2) with variable K can be found in [5], and its efficiency in solving the wave equation (1) on conforming domains is examined in [1].

3 Difference potentials

The method of difference potentials incorporates a given finite difference scheme to solve problems efficiently on nonconforming geometries while maintaining the design convergence rate. For a general domain Ω, we embed the problem within an auxiliary domain Ω_0 which is a simple shape (e.g., a square, as in Figure 2). The shape of Ω_0 along with its boundary conditions should be chosen so that the PDE on Ω_0 is well-posed, but otherwise can be chosen for convenience. The key feature of the method of difference potentials is that the original problem on Ω is reformulated as an equivalent set of problems on the auxilary domain Ω_0 with different right-hand sides.

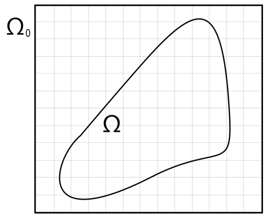

Figure 2: Domain for the method of difference potentials.

4 Time marching with difference potentials on each step

After discretizing the wave equation (1) in time, at time t_n we solve the modified Helmholtz equation (2) with $K = \frac{1}{\theta c^2 h_t^2}$ on Ω by difference potentials, where the right-hand side $G = G(x, y, t_n)$ on Ω is given. In 2D, the auxiliary problem is given by the modified Helmholtz equation (2) on the auxiliary domain Ω_0 which is a square with homogeneous Dirichlet boundary conditions.

Three finite difference solves on the auxiliary domain Ω_0 are required at each time step to produce the solution on the nonconforming domain Ω with high order accuracy, with the right-hand sides determined by the method of difference potentials. The solutions of the resulting finite difference problems on Ω_0 can be computed efficiently by iterative methods.

Fourth order convergence in both space and time for Dirichlet and Neumann problems has been demonstrated using $\theta = \frac{1}{12}$ for variable wave speeds on a nonconforming disk centered at the origin in 2D (Figure 3).

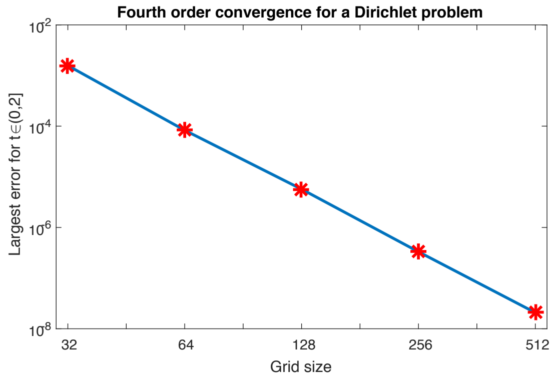

Figure 3: Variable wave speed $c(r) = \frac{r}{4} + 1$ on a nonconforming disk with $CFL = 0.6$. Error is measured from the test solution $u = \cos(5x)\cos(2y)\cos(4t)$.

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