High-order numerical solution of the Helmholtz equation for domains with reentrant corners

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Abstract

Standard numerical methods often fail to solve the Helmholtz equation accurately near reentrant corners, since the solution may become singular. The singularity has an inhomogeneous contribution from the boundary data near the corner and a homogeneous contribution determined by boundary conditions far from the corner. We present a regularization algorithm that uses a combination of analytical and numerical tools to distinguish between these two contributions and ultimately subtract the singularity. We then employ the method of difference potentials to numerically solve the regularized problem with high-order accuracy on a domain with a curvilinear boundary. Our numerical experiments show that the regularization successfully restores the design rate of convergence.

Keywords: singular solutions, regularization, difference potentials

We consider the constant coefficient homogeneous Helmholtz equation on a bounded 2D domain with a reentrant corner, see Figure 1. The PDE is supplemented with Dirichlet boundary conditions on each segment of the boundary:

$$\Delta u + k^2 u = 0 \quad \text{on } \Omega, \tag{1a}$$

$$u\big|_{\Gamma_1} = \varphi_1, \quad u\big|_{\Gamma_2} = \varphi_2, \quad u\big|_{\Gamma_3} = \varphi_3. \tag{1b}$$

Problems with reentrant corners are hard because the solution may become singular near the corner, i.e., the derivatives of the solution become unbounded. Standard numerical methods perform poorly near singularities, so they must be modified before use on singular problems. Wave problems with reentrant corners may arise, for instance, when analyzing the scattering of radar waves near an air–ocean–sea ice interface. Marin et al. [1] have solved several Helmholtz-type equations on domains with reentrant corners using BEM and the method of fundamental solutions.

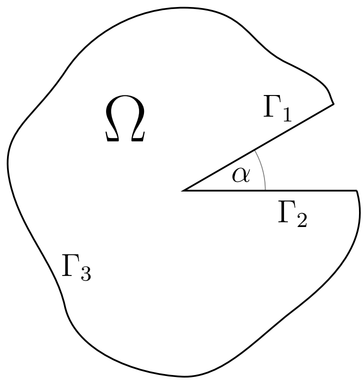

Figure 1: A schematic for the domain Ω with a reentrant corner.

We use regularization (i.e., singularity subtraction) and the method of difference potentials [3] to achieve high-order accuracy near a corner. Singular solutions to the boundary value problem that are expected to hamper numerical convergence are first subtracted out to produce a regularized problem, whose solution is known ahead of time to be smooth enough to be solved numerically without loss of accuracy. The regularized problem is then solved numerically with the method of difference potentials.

The key difficulty with this problem is that there may be two contributions to the singularity which must be handled individually. If we temporarily ignore the boundary condition on the outer boundary Γ_3, we can write the solution u to the Helmholtz equation over the domain Ω as $u = v + w$, where v is a particular solution that satisfies the boundary conditions on the sides of the wedge and

$$w(r,\theta) = \sum_{m=1}^{\infty} a_m J_{m\nu}(kr) \sin(m\nu(\theta - \alpha))$$

is an arbitrary linear combination of solutions that satisfy the homogeneous boundary conditions. Both the particular solution v and the Fourier–Bessel series w may be singular, and we refer to these two components of the singularity as the inhomogeneous contribution and homogeneous contribution, respectively. The inhomogeneous contribution is local, in the sense that it is determined by the boundary conditions in the vicinity of the corner. We use the methodology of Fox and Sankar [2] to derive an asymptotic series for v near the corner:

$$v(r,\theta) \sim \sum_{m=1}^{\infty} v^{(m)}(r,\theta) \quad (r \to 0).$$

The work [2] provides a constructive procedure for determining the terms $v^{(m)}$ from the Helmholtz equation and boundary conditions near the corner. These terms have increasing regularity, as do the Bessel functions $J_{m\nu}$, so we propose the regularization

$$\begin{aligned} u = u^{(\text{reg})} + v^{(1)} + \ldots + v^{(M_v)} \\ + \sum_{m=1}^{M_w} a_m J_{m\nu}(kr)\sin(m\nu(\theta-\alpha)), \end{aligned} \tag{2}$$

where the fixed integers M_v and M_w are chosen large enough to guarantee that $u^{(\text{reg})}$ has a certain number of bounded derivatives.

Unlike the inhomogeneous contribution, the homogeneous contribution is nonlocal, since the unknown intensity factors (a_m) that characterize w are determined by the boundary condition on Γ_3, far from the corner. To compute the leading intensity factors $a_1, \ldots, a_{M_w}$ for use in the regularization (2), we must know what portion of the boundary data on Γ_3 is from w, and what portion is from v. When both v and w are nonzero, "splitting" the data on Γ_3 becomes a challenging issue. In this way, our work is more general than that of Marin et al. [1], who have considered problems with only homogeneous contributions to the singularity. Once the leading intensity factors are computed, the original boundary value problem (1) is recast in terms of the sufficiently smooth function $u^{(\text{reg})}$ to form the regularized problem.

The method of difference potentials [3] uses the discrete counterparts of Calderon's operators to accommodate general curvilinear boundaries while leveraging the accuracy and efficiency of high-order finite difference schemes. This way, the method of difference potentials overcomes a primary limitation of finite difference methods, their inability to accurately handle boundaries that do not conform to the

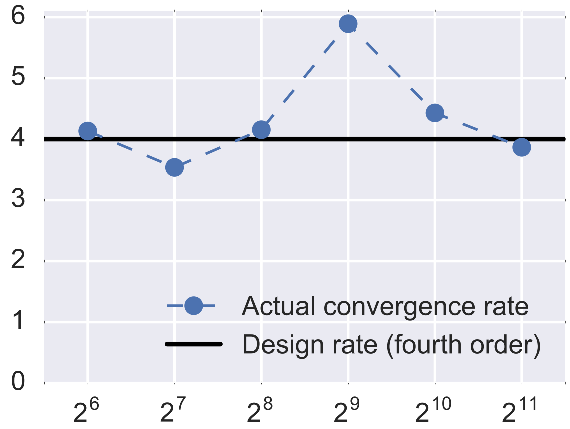

Figure 2: Convergence rate vs. grid dimension.

discretization grid. The method of difference potentials has the same asymptotic complexity as finite difference schemes on regular structure grids. In FEM, on the other hand, high-order accurate approximations can be built for arbitrarily shaped boundaries only in fairly sophisticated and costly algorithms with isoparametric elements.

We have applied the method of difference potentials to the regularized problem for several different configurations of the boundary and data. In all cases we found that the regularization restored the design fourth order convergence; see Figure 2 for the results from one such experiment. Future work could extend our methodology to more difficult cases, such as time-dependent waves, or reentrant corners that lay on the interface between two materials.

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