Lacunae-Based Artificial Boundary Conditions for the Numerical Simulation of Unsteady Waves Governed by Vector Models*

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Abstract. Artificial boundary conditions (ABCs) are constructed for the computation of unsteady acoustic and electromagnetic waves. The waves propagate from a source or a scatterer toward infinity, and are simulated numerically on a truncated domain, while the ABCs provide the required closure at the external artificial boundary. They guarantee the complete transparency of this boundary for all the outgoing waves. They are non-local in both space and time but can be implemented efficiently because their temporal non-locality is fixed and limited. The restriction of temporal nonlocality of the proposed ABCs does not come as a result of any model simplification or approximation, but rather as a consequence of a fundamental property of the solutions — the presence of lacunae, or in other words, sharp aft fronts of the waves, in odd-dimension spaces.

1 Outline of the Algorithm Properties

Two major well-recognized difficulties encountered when computing the propagation of waves over unbounded domains are the accumulation of error during long time intervals and the necessity to truncate the domain and subsequently set the artificial boundary conditions (ABCs) as a closure for the resulting finite formulation. Our previous work conducted for the scalar wave equation indicates that the two aforementioned issues are closely related, [4,5]. Namely, we have used an inherently three-dimensional phenomenon of lacunae, which amounts to the presence of sharp aft fronts of the waves in the solutions of the Cauchy problem, and developed a methodology that modifies any appropriate finite-difference scheme for the wave equation so that the long-term error buildup is eliminated. At the same time, all original properties of the underlying scheme (foremost, its order of accuracy) are fully preserved. More precisely, for a problem of radiation of waves by a continuously operating source, which is compactly supported in space for all times, the algorithm guarantees temporally uniform grid convergence of the solution on a finite computational domain that fully contains the source region. The rate of convergence is the same as that of the original scheme.

The key idea of taking advantage of lacunae is simple. As the propagation speed is finite and the computational domain size is finite, any wave originating

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inside the domain will leave it completely no later than after a finite interval of time that is determined only by the geometry and the speed. With the appropriate partition of the continuously operating source that drives the problem into a collection of finite-duration sources, the solution at any given moment of time can be represented as a finite sum of terms that each needs to be integrated numerically only over a finite fixed time interval. Both aforementioned quantities, the number of terms and the interval of integration, remain fixed and bounded for all times, which easily translates into temporally uniform grid convergence estimates. Moreover, during the same time interval that is needed for a given wave to cross the domain, no other wave can propagate in any other direction beyond a certain distance in space that is again determined by the propagation speed. This implies that the entire computation can only be conducted on a bounded auxiliary domain.

The introduction of the latter bounded domain facilitates the construction of a finite-dimensional discretization and as such, leads to obtaining highly accurate non-local unsteady ABCs for a class of combined problems that may include complex phenomena on a given interior region but reduce to the homogeneous wave equation in the far field, [3,6]. These ABCs are built directly for the specific discrete approximation used in the interior, and can be considered its most natural extension. In other words, the procedure of obtaining the lacunae-based ABCs "bypasses" the two steps common for other existing algorithms — rational approximation of the so-called non-reflecting kernels and discretization of the continuous boundary conditions. This substantially reduces the chances of encountering instabilities, in particular, long-term instabilities. Most important, the extent of temporal nonlocality of the lacunae-based ABCs appears fixed and bounded for all times. This bound is not a consequence of any approximation, it rather follows from the fundamental properties of the solutions that satisfy the Huygens' principle.

Subsequently, the lacunae-based algorithm has been extended to include the cases of waves governed by systems of PDEs, or in physical terminology, by vector models. The two key models considered in this context are acoustics and electrodynamics. The waves are again assumed to propagate from a given bounded region of space outward. The mechanism of wave generation inside this region is not of a particular concern. The waves can be produced by an actual source of any specific (complex) nature or by a scatterer. What is important, that beyond this region, i.e., in the far field, the propagation be governed by a linear constant-coefficient system, see. e.g., equations (1) or (2) below. The properties of the ABCs remain the same — they are built directly for the discretization, they guarantee the complete transparency of the external artificial boundary for all the outgoing waves, and the extent of their temporal nonlocality is fixed and limited, which comes as a natural consequence of the lacunae in the solution.

The proposed ABCs for acoustics and electromagnetics are very versatile. They can be built for any consistent and stable finite-difference scheme. Their accuracy can always be made at least as high as that of the interior solver. Moreover, because of the lacunae this accuracy will not deteriorate even when integrating over long time intervals. The ABCs are also very flexible geometrically and can handle artificial boundaries of irregular shape on regular grids with no fitting/adaptation needed and no accuracy loss induced. Finally, they allow for a wide range of model characteristics to be taken into account. In particular, in the context of sound propagation one can analyze not only the standard ambient acoustics case and the simplest advective acoustics case with uniform background flow, but also the case when the waves' source or scatterer is engaged in an accelerated motion (e.g., a maneuvering aircraft). The latter setup has always been considered difficult to treat. To the best of our knowledge, no successful attempt of constructing the corresponding ABCs has been done previously. In the context of propagation of electromagnetic waves, the capability of handling the non-stationary sources, especially those that move with acceleration, is apparently somewhat less of an issue than in acoustics. Still, there may be applications for which this capability is important, e.g., in astrophysics.

The actual vector models that we have analyzed in the current study, both theoretically and numerically, were the two-dimensional unsteady acoustics equations (linearized Euler) and two-dimensional Maxwell's equations, both with cylindrical symmetry: (r, z, t). Cylindrical symmetry obviously allows us to take full advantage of the crucial three-dimensional lacunae effects in an essentially two-dimensional computational setting. In acoustics, we therefore assume that the velocity vector has no angular component, and that no quantity in the model depends on the polar angle θ, which yields:

$$\begin{aligned}
&\frac{1}{c^2}\frac{\partial p}{\partial t} + \frac{1}{r}\frac{\partial ru}{\partial r} + \frac{\partial w}{\partial z} = 0,\\
&\frac{\partial u}{\partial t} + \frac{\partial p}{\partial r} = 0,\\
&\frac{\partial w}{\partial t} + \frac{\partial p}{\partial z} = 0,
\end{aligned} \tag{1}$$

where c is the speed of sound and p, u, and v are acoustic pressure, radial velocity, and axial velocity, respectively. For the Maxwell equations, we analyze the appropriate transverse magnetic (TM) mode with respect to the radial and axial components of the magnetic field: H_r and H_z, and the angular component of the electric field E_θ:

$$\begin{aligned}
&\frac{1}{c}\frac{\partial E_\theta}{\partial t} = \frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r},\\
&\frac{1}{c}\frac{\partial H_r}{\partial t} = \frac{\partial E_\theta}{\partial z},\\
&\frac{1}{c}\frac{\partial H_z}{\partial t} = \frac{1}{r}\frac{\partial r E_\theta}{\partial r},
\end{aligned} \tag{2}$$

where c is the speed of light. System (2) satisfies a particular gauge (see, e.g., [1]): $\varphi = 0$, $E_\theta = -\frac{1}{c}\frac{\partial A_\theta}{\partial t}$, $H_r = -\frac{\partial A_\theta}{\partial z}$, $H_z = \frac{1}{r}\frac{\partial r A_\theta}{\partial r}$, where φ is the scalar potential and $\boldsymbol{A} = (A_r, A_\theta, A_z)$ is the vector potential. We re-emphasize that the linear homogeneous constant-coefficient systems (1) and (2) only govern the propagation of waves in the far field. As such, they are employed to construct the far-field ABCs, whereas the actual mechanism of waves' generation in the near field may be substantially more complex. We also note that systems (1) and (2) look similar but not quite identical. The differences are obviously accounted for by the vector nature of all the quantities involved in the Maxwell system. Comparison between these two systems is instrumental for understanding geometry of the lacunae, as well as sufficient conditions for their existence. The latter, in turn, utilize existence of the potential.

2 Numerical Demonstrations

On the computer, each of the systems (1) and (2), is integrated by an appropriate second-order central-difference staggered scheme, which can be considered a flavor of the well-known Yee scheme [7]. No reduction of either system to a set of independent wave equations is required. We compare our numerical solutions with the analytic solutions driven by some specially constructed sources. These solutions have been obtained using the most general form of retarded potentials that takes into account the non-uniform motion of the wave sources, see [2]. Comparison with the exact solution allows us to experimentally demonstrate the design convergence rate of the algorithm over long time intervals.

In Figure 1, we show the logarithmic error curves for the acoustic pressure. The source cyclically speeds up and slows down along the z axis with the minimum speed 0 and maximum speed $0.2c$. It also generates periodic waves with the frequency three times that of the motion oscillations. The computation was

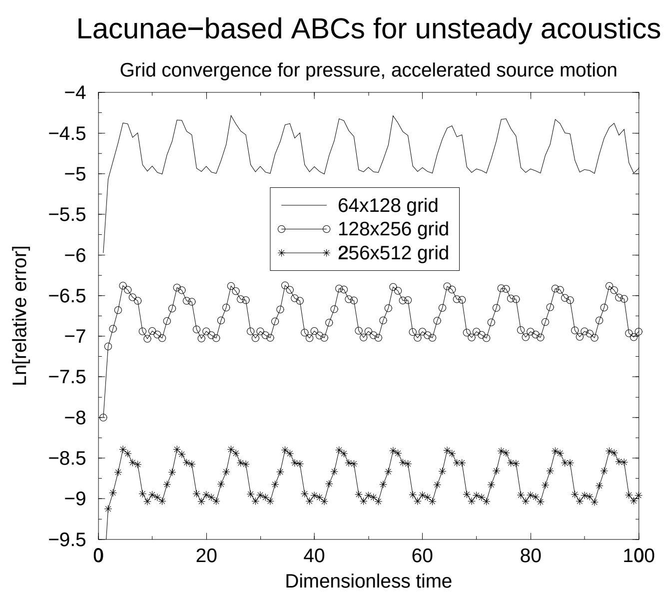

Fig. 1. Second-order central-difference scheme for cylindrically symmetric acoustics.

continued till $t = 100$, which is one hundred times the interval required for the waves to cross the domain. Figure 1 clearly indicates the second order of grid convergence. The convergence also does not deteriorate with time. Similar results for the Maxwell equations are presented in Figure 2.

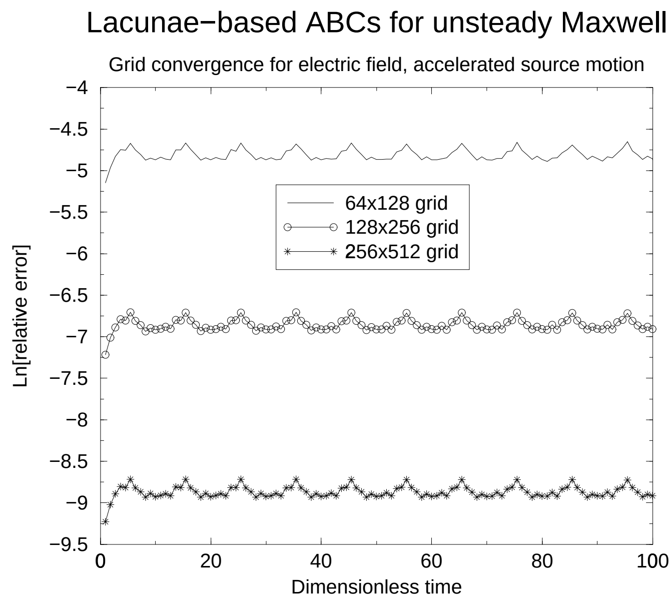

Fig. 2. Second-order central-difference scheme for cylindrically symmetric Maxwell.

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