## ARTIFICIAL BOUNDARY CONDITIONS FOR THE NUMERICAL SOLUTION OF EXTERNAL VISCOUS FLOW PROBLEMS\*

V. S. RYABEN'KII<sup>†</sup> AND S. V. TSYNKOV<sup>‡</sup>

**Abstract.** In this paper we describe an algorithm for the nonlocal artificial boundary conditions setting at the external boundary of a computational domain while numerically solving unbounded viscous compressible flow problems past the finite bodies. Our technique is based on the usage of generalized Calderon projection operators and the application of the difference potentials method. Some computational results are presented.

Key words. artificial boundary conditions, boundary equations with projections, auxiliary problem, difference potentials method

AMS subject classifications. 65N99, 76M25

1. Introduction. The numerical solution of external boundary-value problems usually requires the application of special procedures for adequate consideration of the solution structure in the whole unbounded domain. The need for developing such special procedures is due to the computer limitations. At present it is possible to point out two different approaches. The first one deals with the singular coordinate transformations realizing one-to-one mappings between the unbounded original domain and the new finite domain. The second one is based on the so-called artificial boundaries introduction and artificial boundary conditions (ABCs) setting. Following such a technique, one ought only to compute a solution in some finite subdomain of the original domain. Special conditions at the boundary of a subdomain (this boundary is called an artificial) are to be formulated in such a way as to provide maximal proximity (in a certain sense) of the solution obtained in the finite subregion to the corresponding fragment of the original problem solution. Generally speaking, ABCs can be used not only in the case of an unbounded original domain but also when it is simply sufficient (for any reason) to know the solution of the problem not everywhere but only in some subdomain of the original domain. A detailed review and comparison of different well-known techniques of ABCs' construction are given in [1], [2], including some applications to the problems of elasticity, acoustics, fluid dynamics, waves propagation, etc. Some review information is also contained in [3].

In the current paper the ABCs for the numerical solution of the external viscous flow problems are developed; that is, we consider an unbounded viscous compressible gas flow over the finite body in the stationary two-dimensional (2D) (plane) formulation. The choice of geometry, problem dimensionality, and free stream parameters (uniform subsonic flow) is not caused by any fundamental restrictions, but only because the theoretical analysis is apparently the least cumbrous in such a case and the computational resources required are not very large.

The following assumption is the principle for our technique of ABCs' construction: flow perturbations caused by the immersed body are small far enough from it and,

<sup>\*</sup> Received by the editors February 8, 1993; accepted for publication (in revised form) January 31, 1994.

<sup>&</sup>lt;sup>†</sup> Keldysh Institute of Applied Mathematics, Russian Academy of Sciences, Miusskaya sq. 4, 125047 Moscow, Russia (**ryab@applmat.msk.su**). The work of this author was partially supported by the Russian Fund for Fundamental Research project 93-012-859.

<sup>&</sup>lt;sup>‡</sup> Department of Applied Mathematics, School of Mathematical Sciences, Tel-Aviv University, Ramat-Aviv, 69978 Tel-Aviv, Israel. Current address: Mail Stop 128, 18e West Taylor Street, NASA Langley Research Center, Hampton, Virginia 23681 (s.v.tsynkov@larc.nasa.gov).

consequently, the governing equations can be considered as linear in the far field.

Let us clarify here how we understand the concept of linearization. Assume that A is a finite domain in  $\mathbb{R}^2$  (it is an immersed body hereafter) and we solve the problem

$$\mathbf{F}_x \mathbf{u} = \mathbf{0}$$

in  $\overline{R^2 \setminus A}$ . Here  $\mathbf{F}_x$  is, generally speaking, a nonlinear differential operator and the subscript "x" underlines the possibility to compute its action locally in each point  $x \in \overline{R^2 \setminus A}$ ;  $\mathbf{u} \in U$  and U is some space of (vector-)functions where the solution is to be found. Boundary conditions at  $\partial A$  and while  $|x| \longrightarrow +\infty$  are included in the definitions of  $\mathbf{F}_x$  and U. We consider the full Navier–Stokes equations (second-order nonlinear system of four equations) as (1.1) hereafter (see §2 and further) with the nonslip conditions at  $\partial A$  and  $\mathbf{u} \longrightarrow \mathbf{u}_0$  condition while  $|x| \longrightarrow +\infty$ ; here  $\mathbf{u}_0$  are the free stream parameters (for more details about the latter condition see also §2 and further).

Choose now two subdomains  $D_{\text{in}}$  and  $D_{\text{ex}}$  in  $\overline{R^2 \setminus A}$  such that  $\overline{D_{\text{in}} \cup D_{\text{ex}}} = \overline{R^2 \setminus A}$ ,  $D_{\text{in}}$  is bounded and completely surrounds A, i.e.,  $\{\forall x' \in \partial A : x' \in \delta D_{\text{in}}; \exists x'' \in \partial D_{\text{in}}: x'' \notin \partial A; \text{ dist}(A, \partial D_{\text{in}} \setminus \partial A) > 0; D_{\text{in}} \cup A \text{ is a simply connected domain}\},$ and  $D_{\text{ex}} \stackrel{\text{def}}{=} R^2 \setminus \overline{D_{\text{in}} \cup A}$  is unbounded. We will name the sufficiently smooth non-selfcrossing curve  $\Gamma \stackrel{\text{def}}{=} \partial D_{\text{ex}} = \partial D_{\text{in}} \setminus \partial A$  an artificial boundary. Replace the nonlinear equation (1.1) in  $D_{\text{ex}}$  by the linear one using the Frechet derivative  $\mathbf{F}' [\mathbf{u}_0]_x$  of operator  $\mathbf{F}_x$  in the point  $\mathbf{u}_0 \in U$  (the action of operator  $\mathbf{F}' [\mathbf{u}_0]_x$  is also computed by means of local formulae). Then we pass to the following system:

(1.2) 
$$\mathbf{F}_{x}\mathbf{u}_{0} + \mathbf{F}' [\mathbf{u}_{0}]_{x} (\mathbf{\check{u}} - \mathbf{u}_{0}) = \mathbf{0}, \quad x \in D_{\mathrm{ex}},$$
$$\mathbf{F}_{x}\mathbf{\check{u}} = \mathbf{0}, \quad x \in \overline{D}_{\mathrm{in}}$$

to be solved with respect to the unknown function  $\check{\mathbf{u}} \in U$ . Note that if  $D_{\text{ex}}$  coincides with  $R^2 \setminus A$  (and, consequently,  $D_{\text{in}} = \emptyset$ ) then (1.2) is the first iteration of the operator Newton method

$$\mathbf{u}_{s+1} = \mathbf{u}_s - \left(\mathbf{F}' \left[\mathbf{u}_s\right]_x\right)^{-1} \mathbf{F}_x \mathbf{u}_s, \quad s = 1, 2, \dots, \ x \in \mathbb{R}^2 \setminus \mathbb{A}$$

described, e.g., in [4].

We will try to get a sufficiently accurate approximate solution  $\check{\mathbf{u}}$  from (1.2) using the freedom existing in the choice of artificial boundary  $\Gamma$  location, i.e., in the choice of form and size of  $D_{\rm in}$  (and  $D_{\rm ex}$ ). The error involved in  $\check{\mathbf{u}}$  is evidently caused by the replacement of nonlinear equation (1.1) in  $D_{\rm ex}$  by the linear one. This error is small when the exact solution  $\mathbf{u}$  itself of equation (1.1) slightly deviates from the background  $\mathbf{u}_0$ ; i.e., when it takes place,

(1.3) 
$$\frac{|\mathbf{\check{u}}(x) - \mathbf{u}_0|}{|\mathbf{u}_0|} \ll 1.$$

Assumption (1.3) (see §2 for more details) is quite naturally far enough from the immersed body (i.e., in the domain  $D_{\text{ex}}$  if  $\operatorname{dist}(A, \Gamma)$  is sufficiently large). However, we will be interested not only in the smallness itself of the error introduced into the solution by linearization, but also in the possibility of control of this error value, i.e., in the possibility of making the solution  $\mathbf{\check{u}}$  more precise. Substituting this solution into (1.1) we get the residual  $\mathbf{F}_x \mathbf{\check{u}} = \mathbf{0}, x \in \overline{D}_{\text{in}}$ , and  $\mathbf{F}_x \mathbf{\check{u}} = \Delta \mathbf{F} [\mathbf{u}_0]_x (\mathbf{\check{u}} - \mathbf{u}_0), x \in D_{\text{ex}}$ . Here  $\Delta \mathbf{F} [\mathbf{u}_0]_x$  is a nonlinear remainder term, i.e.,

 $\forall \mathbf{u}: \mathbf{F}_x \mathbf{u} = \mathbf{F}_x \mathbf{u}_0 + \mathbf{F}' [\mathbf{u}_0]_x (\mathbf{u} - \mathbf{u}_0) + \Delta \mathbf{F} [\mathbf{u}_0]_x (\mathbf{u} - \mathbf{u}_0)$ . It would be possible to use some iteration procedure starting from  $\check{\mathbf{u}}$  to obtain the exact solution  $\mathbf{u}$  of equation (1.1). In such case the residual of the previous iteration determines the contribution of the next one to the solution.

The following condition

(1.4) 
$$\mathbf{\check{u}} - \mathbf{u}_0 \longrightarrow \mathbf{0}, \quad |x| \longrightarrow +\infty$$

should evidently be valid for the function  $\mathbf{\check{u}}$  itself. Therefore, *if it occurs* that the value of the residual  $\Delta \mathbf{F} [\mathbf{u}_0]_x (\mathbf{\check{u}}(x) - \mathbf{u}_0)$  vanishes while  $|x| \longrightarrow +\infty$  faster than  $\mathbf{\check{u}}(x) - \mathbf{u}_0$  itself, i.e., if

(1.5) 
$$\frac{|\Delta \mathbf{F}[\mathbf{u}_0]_x \left(\mathbf{\check{u}}(x) - \mathbf{u}_0\right)|}{|\mathbf{\check{u}}(x) - \mathbf{u}_0|} \longrightarrow 0, \quad |x| \longrightarrow +\infty,$$

then it means that the relative contribution of the next iteration to the solution becomes smaller while receding from the body A and, consequently, the function  $\tilde{\mathbf{u}}$ better approximates the exact solution. Therefore, in the case of the validity of (1.5) it is possible in principle to make the approximate solution  $\tilde{\mathbf{u}}$  from (1.2) asymptotically more precise by means of the size of  $D_{\text{in}}$  enlargement. In other words, locating the artificial boundary  $\Gamma$  at sufficient distance from A, we can get the solution  $\tilde{\mathbf{u}}$  from (1.2) that approximates the solution  $\mathbf{u}$  of the original problem (1.1) with desirable accuracy.

Thus, we treat linearization as a replacement of (1.1) by (1.2) yielding a sufficiently good (i.e., sufficiently close to the solution **u** of equation (1.1)) approximation  $\tilde{\mathbf{u}}$  from (1.2) on the basis of (1.3) and, moreover, enabling the approximate solution  $\tilde{\mathbf{u}}$  to become asymptotically more precise by means of  $D_{\rm in}$  size enlargement.

In this paper we do not consider general questions about linearization permissibility; i.e., for what flow regimes (defined by the Mach and Reynolds numbers in the first turn) does such a subregion of the original domain exist where (1.3) and (1.5) are valid, and how far from the body A can one place the artificial boundary  $\Gamma$ to obtain satisfactory accuracy of the approximation? These questions require some additional investigation, both theoretically and by means of numerical experiments. However, one can assume that at least for the low Ma and Re numbers, linearization is possible; moreover, the configuration of domains  $D_{in}$  and  $D_{ex}$  corresponds to the one described above, i.e., the domain  $D_{in}$ , in which we use original nonlinear equation (1.1), is finite. Our numerical experiments (§6) justify this assumption.

The fundamental difference between (1.2) and (1.1) is just that we need to solve the nonlinear equation in (1.2) only in some finite domain. It is easy to see that the nonlinear problem in  $D_{in}$  and the linear one in  $D_{ex}$  (see (1.2)) are not independent and ought to be solved concurrently. But we will replace the whole linear part by the equivalent ABCs at the boundary  $\Gamma$ . Then the solution of the nonlinear problem in  $D_{in}$  with these ABCs will coincide with the solution of the whole coupled problem (1.2).

Our technique of ABCs' construction is not connected with any specific method of numerical integration of the Navier–Stokes equations inside the finite computational domain  $D_{in}$ . A class of such methods is well known and described in the literature; see, e.g., [5], [6], and the bibliographies there. Therefore we do not go into the procedure of the solution of the nonlinear problem in  $D_{in}$ , but focus on the construction of some special conditions at the artificial boundary. They should be equivalent to the linear differential equation (system) from (1.2) combined with the corresponding condition (1.4) at infinity. These relations will be, generally speaking, spatially nonlocal. They are the operator equations containing generalized Calderon boundary projection operators, the discretization of which is implemented by means of the difference potentials method (DPM) [7], [8]. These boundary operator equations are used as ABCs and are constructed hereafter in the discrete formulation compatible with some finitedifference method used inside  $D_{\rm in}$ . By virtue of the equivalence mentioned above (the linear problem in  $D_{\rm ex} \Leftrightarrow$  boundary relations at  $\Gamma$ ) the ABCs obtained in this way guarantee the possibility of uniquely complementing the solution found in  $\overline{D}_{\rm in}$  to the solution of (1.2) in  $R^2 \backslash A$  provided that the latter exists and is unique.

The material hereafter is prepared as follows. The general scheme of the ABCs' setting based on the concept of problem decomposition into the "linear" and "non-linear" parts and on the application of the generalized Calderon–Seeley projection operators [9], [10] is stated in §2. The constructions of §2 are based on the fundamental concept of the auxiliary problem (AP) (see [7], [8]) formulated there in the infinite strip parallel to the y-axis. In §3 it is shown how to pass to the AP in some finite domain for the discrete case. Section 4 is devoted to the description of a finite-difference algorithm for the solution of the AP. Section 5 is devoted to the construction of the difference ABCs; and, finally, some numerical results are described in §6. Section 7 is the Appendix.

Notice, in addition, that an analogous technique for the inviscid Euler flow is developed and implemented numerically in [3], [11].

2. Continual ABCs. Consider the plane stationary flow of viscous compressible perfect gas governed by the Navier–Stokes equations:

(2.1) 
$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0,$$

$$\begin{split} \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} &= \frac{1}{Re} \left[ \frac{4}{3} \frac{\partial}{\partial x} \mu \frac{\partial u}{\partial x} - \frac{2}{3} \frac{\partial}{\partial x} \mu \frac{\partial v}{\partial y} + \frac{\partial}{\partial y} \mu \frac{\partial v}{\partial x} + \frac{\partial}{\partial y} \mu \frac{\partial u}{\partial y} \right], \\ \rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} &= \frac{1}{Re} \left[ \frac{\partial}{\partial x} \mu \frac{\partial v}{\partial x} + \frac{\partial}{\partial x} \mu \frac{\partial u}{\partial y} - \frac{2}{3} \frac{\partial}{\partial y} \mu \frac{\partial u}{\partial x} + \frac{4}{3} \frac{\partial}{\partial y} \mu \frac{\partial v}{\partial y} \right], \\ \rho u \frac{\partial \varepsilon}{\partial x} + \rho v \frac{\partial \varepsilon}{\partial y} + p \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = \frac{1}{Re} \left[ -\frac{2}{3} \mu \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)^2 \right] \\ + \mu \left( 2 \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)^2 + 2 \left( \frac{\partial v}{\partial y} \right)^2 \right) + \frac{\gamma}{Pr} \left( \frac{\partial}{\partial x} \mu \frac{\partial \varepsilon}{\partial x} + \frac{\partial}{\partial y} \mu \frac{\partial \varepsilon}{\partial y} \right) \right], \\ \varepsilon &= \frac{1}{\gamma - 1} \frac{p}{\rho} \qquad \text{equation of state;} \end{split}$$

here (x, y) are Cartesian coordinates, (u, v) — Cartesian projections of velocity vector, p — pressure,  $\rho$  — density,  $\varepsilon$  — internal energy,  $\mu$  — viscosity coefficient,  $\gamma$ — specific ratio,  $Re = (\rho_0 u_0 L)/\mu_0$  — Reynolds number, L — characteristic size,  $Pr = (\mu c_p)/\kappa$  = const — Prandtl number,  $\kappa$  — heat conduction coefficient. Subscript "0" denotes free stream parameters as before. System (2.1) is written just in the dimensionless form, the value  $u_0$  is chosen as the velocity scale, and we'll also assume hereafter that the free stream is directed from the left to the right parallel to the x-axis, so  $v_0 = 0$ ,  $|u_0| = u_0$ ,  $\rho_0$  is the density scale,  $\rho_0 (u_0)^2$  is the pressure scale,  $\mu_0$  is the viscosity scale,  $(u_0)^2$  is the internal energy scale.

Assume now that the deviations of the flow parameters (noted by a tilde "~" over the corresponding letter hereafter) are small in the domain  $D_{\text{ex}}$ . It means for the dimensionless values

(2.2) 
$$\rho = 1 + \tilde{\rho}, \ u = 1 + \tilde{u}, \ v = \tilde{v}, \ \mu = 1 + \tilde{\mu}, \ p = \left(\gamma(Ma)^2\right)^{-1} + \tilde{p},$$
$$\varepsilon = \left((\gamma - 1)\gamma(Ma)^2\right)^{-1} + \tilde{\varepsilon};$$
$$\tilde{\rho} \ll 1, \ \tilde{u} \ll 1, \ \tilde{v} \ll 1, \ \tilde{\mu} \ll 1, \ \tilde{p} \ll \left(\gamma(Ma)^2\right)^{-1},$$
$$\tilde{\varepsilon} \ll \left((\gamma - 1)\gamma(Ma)^2\right)^{-1},$$

where Ma is the free stream Mach number  $Ma = u_0 \left(\gamma \left(p_0/\rho_0\right)\right)^{-1/2}, Ma < 1.$ Conditions (2.2) are the specific definition of (1.3).

Substituting (2.2) into (2.1) and retaining only the linear terms with respect to small deviations, we obtain (the tilde " $\sim$ " is omitted everywhere because henceforth we'll deal only with the equations "in perturbations")

(2.3) 
$$\frac{\partial \rho}{\partial x} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

$$\frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} - \frac{1}{Re} \left[ \frac{4}{3} \frac{\partial^2 u}{\partial x^2} + \frac{1}{3} \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} \right] = 0,$$

$$\frac{\partial v}{\partial x} + \frac{\partial p}{\partial y} - \frac{1}{Re} \left[ \frac{4}{3} \frac{\partial^2 v}{\partial y^2} + \frac{1}{3} \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x^2} \right] = 0,$$

$$\frac{\partial p}{\partial x} - \frac{1}{(Ma)^2} \frac{\partial \rho}{\partial x} - \frac{\gamma}{Re Pr} \left[ \Delta p - \frac{1}{\gamma (Ma)^2} \Delta \rho \right] = 0$$

The equation of state was used for  $\varepsilon$  elimination while deriving the last equation (2.3);  $\Delta \equiv (\partial^2/\partial x^2) + (\partial^2/\partial y^2)$  is the Laplace operator in (2.3). In addition, note that the term corresponding to  $\mathbf{F}_x \mathbf{u}_0$  from the first equation in (1.2) turns into zero here.

The vanishing of all the unknown variables (see (1.4)) is the boundary condition at infinity for (2.3):

(2.4) 
$$\rho \longrightarrow 0, \ u \longrightarrow 0, \ v \longrightarrow 0, \ p \longrightarrow 0 \text{ while } x^2 + y^2 \longrightarrow \infty,$$

which simply corresponds to the free stream limit of the solution.

Let us now consider (without loss of generality) some strip  $D^0 = \{(x, y) | 0 < x < X\}$  completely containing  $D_{in}$ . Designate  $D = D_{ex} \cap D^0$ . Since (2.3) is the system with constant coefficients it is possible to seek its solution by means of variables' separation. We will formulate and solve an AP in  $D^0$  for the nonhomogeneous version of system (2.3). The boundary conditions for AP are to be set at the boundary of  $D^0$ , i.e., at the straight lines x = 0 and x = X as well as for  $y \longrightarrow \pm \infty$ . They should be equivalent to (2.4) or, in other words, one should be able to smoothly and uniquely complement on  $R^2/D^0$  the solution of AP obtained in  $D^0$  so that it satisfies (2.3)–(2.4) in  $D_{ex}$ .

We will also demand that an AP has a unique solution for any compactly supported right-hand side.

Introduce the spaces  $U^0$  and  $F^0$  of the solutions and the right-hand sides of AP, respectively; as it will be seen later, it is always possible to consider only functions with compact support (belonging to  $D^0$ ) as the right-hand sides of AP while performing specific computations. The elements of these spaces  $\mathbf{u}^0 \equiv (u^0, v^0 p^0, \rho^0)^T \in U^0$ and  $\mathbf{f}^0 \equiv (f_1^0, f_2^0, f_3^0, f_4^0)^T \in F^0$  are the vector functions defined on  $\overline{D^0}$  and  $D^0$ , respectively; the functions  $\mathbf{f}^0$  have compact support:  $\operatorname{supp} \mathbf{f}^0 \subset D^0$ . Moreover, we will include the boundary conditions of AP in the definition of  $U^0$ . First, let the function  $\mathbf{u}^0(x, y) \in U^0$  be bounded on  $D^0$ , absolutely integrable and representable in the form of the Fourier integral along y for all  $x \in (-\infty, +\infty)$ :

(2.5) 
$$\mathbf{u}^0(x, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\mathbf{u}}^0(x, \alpha) e^{i\alpha y} d\alpha.$$

In particular, this implies the vanishing of the solution along each line x = const,  $\mathbf{u}^0(x, y) \longrightarrow 0$  as  $y \longrightarrow \pm \infty$ . We will impose boundary conditions at x = 0 and at x = X just on the Fourier transform  $\hat{\mathbf{u}}^0(x, \alpha)$  for all  $\alpha$ . For this purpose, one first ought to separate variables. Substituting (2.5) into the equation  $\mathbf{L}^0\mathbf{u}^0 = \mathbf{f}^0$ , where  $\mathbf{L}^0$ :  $U^0 \longrightarrow F^0$  is the linear operator from the left-hand side of (2.3), and doing the Fourier transform for  $\mathbf{f}^0$  ( $\mathbf{f}^0(x, y)$ ) has compact support),  $\mathbf{f}^0(x, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\mathbf{f}}^0(x, \alpha) e^{i\alpha y} d\alpha$ , we get for each  $\alpha$  the second-order system of ordinary differential equations (ODEs) with respect to  $\hat{\mathbf{u}}^0(x, \alpha)$ :

$$(2.6) \qquad \qquad \frac{d\,\hat{\rho}^{0}}{d\,x} + \frac{d\,\hat{u}^{0}}{d\,x} + i\alpha\hat{v}^{0} = \hat{f}_{1}^{0}, \\ \frac{d\,\hat{u}^{0}}{d\,x} + \frac{d\,\hat{p}^{0}}{d\,x} - \frac{1}{Re} \left[ \frac{4}{3} \frac{d^{2}\hat{u}^{0}}{d\,x^{2}} + \frac{i\alpha}{3} \frac{d\,\hat{v}^{0}}{d\,x} - \alpha^{2}\hat{u}^{0} \right] = \hat{f}_{2}^{0}, \\ \frac{d\,\hat{v}^{0}}{d\,x} + i\alpha\hat{p}^{0} - \frac{1}{Re} \left[ -\frac{4}{3}\alpha^{2}\hat{v}^{0} + \frac{i\alpha}{3} \frac{d\,\hat{u}^{0}}{d\,x} + \frac{d^{2}\hat{v}^{0}}{d\,x^{2}} \right] = \hat{f}_{3}^{0}, \\ \frac{d\,\hat{p}^{0}}{d\,x} - \frac{1}{(Ma)^{2}} \frac{d\,\hat{\rho}^{0}}{d\,x} - \frac{\gamma}{Re\,Pr} \left[ \frac{d^{2}\hat{p}^{0}}{d\,x^{2}} - \alpha^{2}\hat{p}^{0} - \frac{1}{\gamma(Ma)^{2}} \left( \frac{d^{2}\hat{\rho}^{0}}{d\,x^{2}} - \alpha^{2}\hat{\rho}^{0} \right) \right] = \hat{f}_{4}^{0}. \end{cases}$$

The coefficients of (2.6) are constant and complex-valued; moreover,  $\forall \alpha$  : supp  $\hat{\mathbf{f}}^0(x, \alpha) \subset (0, X)$ . Now introduce the new variables

(2.7) 
$$\chi(x, \alpha) = \frac{d \hat{u}^0(x, \alpha)}{d x},$$
$$\eta(x, \alpha) = \frac{d \hat{v}^0(x, \alpha)}{d x},$$
$$\zeta(x, \alpha) = \frac{d \hat{p}^0(x, \alpha)}{d x},$$
designate

designate

$$\hat{\mathbf{v}}^{0} = \left(\chi, \, \eta, \, \zeta, \, \hat{u}^{0}, \, \hat{v}^{0}, \, \hat{p}^{0}, \, \hat{\rho}^{0}\right)^{T},$$

$$\hat{\mathbf{g}}^{0} = \left(\hat{f}_{2}^{0}, \, \hat{f}_{3}^{0}, \, rac{Re\,Pr}{\gamma} \hat{f}_{4}^{0} - rac{1}{\gamma(Ma)^{2}} \left(\hat{f}_{1}^{0}
ight)_{x}^{\prime} - rac{3\,Re}{4\,\gamma(Ma)^{2}} \hat{f}_{2}^{0}, \, 0, \, 0, \, 0, \, \hat{f}_{1}^{0}
ight)^{T}$$

and obtain instead of (2.6) the following system of seven first-order differential equations

(2.8) 
$$\frac{d\,\hat{\mathbf{v}}^0(x,\,\alpha)}{d\,x} - \mathbf{Q}(\alpha)\hat{\mathbf{v}}^0(x,\,\alpha) = \hat{\mathbf{g}}^0(x,\,\alpha),$$

where the entries of the matrix  $\mathbf{Q}(\alpha)$  are given by

$$\mathbf{Q}(\alpha) = \begin{bmatrix} \frac{\frac{3}{4}Re - 4i\alpha}{-\frac{i\alpha}{3}} & 0 & \frac{3}{4}Re \\ -\frac{i\alpha}{3} & Re & 0 \\ \frac{1}{\gamma(Ma)^2} \begin{bmatrix} 4i\alpha - \frac{3}{4}Re + \gamma \end{bmatrix} & -\frac{i\alpha}{\gamma(Ma)^2} & \frac{Re\,Pr}{\gamma} - \frac{3Re}{4\gamma(Ma)^2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}$$

$\frac{3}{4}\alpha^2$	0	0	ך 0
0	$\frac{4}{3}\alpha^2$	ilphaRe	0
$rac{3lpha^2}{4\gamma(Ma)^2}$	$\frac{ilpha Re Pr}{\gamma (Ma)^2}$	$\alpha^2$	$-\frac{\alpha^2}{\gamma(Ma)^2}$
0	0	0	0
0	0	0	0
0	0	0	0
0	-ilpha	0	0 ]

To satisfy the condition  $\mathbf{u}^0 \longrightarrow 0$ ,  $x \longrightarrow \pm \infty$  (see (2.4)), we need to find a solution to (2.8) on [0, X] that would be a fragment of the solution to (2.8) defined on the whole set  $(-\infty, +\infty)$  and vanishing while  $x \longrightarrow \pm \infty$ . Let us remember that  $\forall \alpha$ : supp  $\hat{\mathbf{g}}^0(x, \alpha) \subset (0, X)$ ; i.e., equations (2.8) are homogeneous outside (0, X). Therefore the boundary conditions on the left end of the interval (x = 0) are to prohibit those solutions of homogeneous system (2.8) which correspond to  $\Re \lambda_s(\alpha) \leq 0$  (here  $\lambda_s(\alpha), s = 1, \ldots, 7$ , are the eigenvalues of the matrix  $\mathbf{Q}(\alpha)$ ) as these solutions evidently do not vanish while  $x \longrightarrow -\infty$ . Analogously, on the right end (x = X) these conditions are to prohibit the solutions corresponding to  $\Re \lambda_s(\alpha) > 0$  as they grow infinitely while  $x \longrightarrow +\infty$ . The case of  $\Re \lambda_s(\alpha) = 0$  for x = X is considered below in §4. One can write the conditions required in the form

(2.9) 
$$\mathbf{S}^{-}(\alpha)\hat{\mathbf{v}}^{0}(0,\,\alpha) = \mathbf{0}, \quad \alpha \in (-\infty, +\infty),$$
$$\mathbf{S}^{+}(\alpha)\hat{\mathbf{v}}^{0}(X,\,\alpha) = \mathbf{0}, \quad \alpha \in (-\infty, +\infty),$$

where  $\mathbf{S}^{-}(\alpha)$  and  $\mathbf{S}^{+}(\alpha)$  are the special rank-deficient matrices 7 × 7 depending on  $\mathbf{Q}(\alpha)$ , with their ranks equal to the numbers of eigenvalues  $\lambda_s(\alpha)$  with nonpositive and positive real part, respectively. These matrices are given by the following expressions:

(2.10) 
$$\mathbf{S}^{-}(\alpha) = \prod_{\Re \lambda_{s}(\alpha) > 0} \left( \mathbf{Q}(\alpha) - \lambda_{s}(\alpha) \mathbf{I} \right),$$
$$\mathbf{S}^{+}(\alpha) = \prod_{\Re \lambda_{s}(\alpha) \le 0} \left( \mathbf{Q}(\alpha) - \lambda_{s}(\alpha) \mathbf{I} \right);$$

here **I** is an identity matrix. We will describe the matrices  $\mathbf{S}^{-}(\alpha)$  and  $\mathbf{S}^{+}(\alpha)$  in detail later on, while dealing with the finite-difference case in §5 (see also the Appendix).

Thus, assuming that the unique solution  $\mathbf{u}^0 \in U^0$  of the AP,

(2.11) 
$$\mathbf{L}^0 \mathbf{u}^0 = \mathbf{f}^0, \quad \mathbf{u}^0 \in U^0,$$

exists for any  $\mathbf{f}^0 \in F^0$ , we construct this solution by means of the Fourier technique. Consequently, the Green operator of the AP (2.11),  $\mathbf{G}^0 : F^0 \longrightarrow U^0$ , is defined. By virtue of the choice of boundary conditions one can uniquely complement the solution  $\mathbf{u}^0$  of the AP (2.11) found in  $\overline{D^0}$  so that it satisfies (2.3) on  $R^2$ , is bounded on  $R^2$ , and vanishes along the lines x = const and y = const,  $\mathbf{u}^0(x, y) \longrightarrow 0$  as  $y \longrightarrow \pm \infty$  for any  $-\infty < x < +\infty$  and as  $x \longrightarrow \pm \infty$  for any  $-\infty < y < +\infty$  (see Lemma 4.1). Further we will use these properties instead of (2.4) (actually, they are weaker than (2.4)). Here we have to make the following important remark.

*Remark.* Since the Navier–Stokes equations are of mixed order, the construction of boundary conditions to be imposed actually depends on the stream direction at the boundary. In our AP we have inflow at x = 0 and outflow at x = X. If one analyzes boundary conditions of a certain prescribed type, e.g., a general inhomogeneous first-order differential relation as in [12], or a general homogeneous second-order differential relation as in [13], it turns out that the Navier–Stokes equations require four such conditions at inflow and three at outflow. Our approach is somewhat different since the boundary conditions (2.9) connect all the functions  $u^0$ ,  $v^0$ ,  $\rho^0$ ,  $\rho^0$  in one matrix equation and, what is more important, they are spatially nonlocal in physical variables. In doing so, the difference between inflow and outflow boundaries is determined by the structure of  $\lambda_s(\alpha)$  (eigenvalues of  $\mathbf{Q}(\alpha)$ ), which may be easily seen from (2.10). Indeed, even the numbers of eigenvalues with positive and nonpositive real parts can be not equal to each other (these numbers may also depend on  $\alpha$ ), which implies that the matrices  $\mathbf{S}^{-}(\alpha)$  and  $\mathbf{S}^{+}(\alpha)$  (2.10) will have different ranks. A natural question arising here refers to any correlations between the local and nonlocal conditions. Though we did not carry out a special investigation, we believe that certain classes of reasonable rational approximations to (2.9) yielding local conditions in physical space may really result in the set of three separate conditions at outflow and four conditions at inflow.

AP is still formulated in the unbounded domain, though unbounded only in one direction. In the next section it will be shown how to pass to the finite domain while solving AP numerically by means of some difference technique. And now assume that we are able to find the solution of AP, i.e., to compute the operator  $\mathbf{G}^0$  and describe briefly the procedure of constructing the boundary equations with projections [7], [8] and their application to the setting of ABCs. We are not going to give here an accurate basis of the method proposed; we only outline the scheme for the continuous case which will be turned into the specific algorithm below for the difference formulation.

Let us introduce the operators:  $\Theta^0$  — complement of an arbitrary function determined in D by zero in  $D^0 \backslash D$  and  $\Theta$  — restriction of a domain of any function determined in  $D^0$  from  $D^0$  to D.  $\overline{\Theta}^0$  and  $\overline{\Theta}$  are the analogous operators for closed domains and

$$\Theta_B(C) \equiv \begin{cases} 1, & x \in C, \\ 0, & x \in B \setminus C \end{cases}$$

is a characteristic function of the set C where B and C,  $C \subset B$ , are arbitrary sets. Define also the spaces  $F^{0^+} \stackrel{\text{def}}{=} \{ \mathbf{f}^0 \in F^0 \mid \Theta_{D^0}(D) \mathbf{f}^0 \in F^0 \}, F^+ \stackrel{\text{def}}{=} \{ \mathbf{f} \stackrel{\text{def}}{=} \Theta \mathbf{f}^0 \mid \mathbf{f}^0 \in F^0 \}$   $F^{0^+}$ ,  $U^{0^+} \stackrel{\text{def}}{=} \{\mathbf{u}^0 \in U^0 \mid \mathbf{L}^0 \mathbf{u}^0 \in F^{0^+}\}$ ,  $U^+ \stackrel{\text{def}}{=} \{\mathbf{u} \stackrel{\text{def}}{=} \overline{\Theta} \mathbf{u}^0 \mid \mathbf{u}^0 \in U^{0^+}\}$  and the operator  $\mathbf{G} : F^+ \longrightarrow U^+$ ,  $\forall \mathbf{f} \in F^+ \mathbf{G} \mathbf{f} \stackrel{\text{def}}{=} \overline{\Theta} \mathbf{G}^0 \Theta^0 \mathbf{f}$ . Let us also point out that the operator  $\mathbf{L} : U^+ \longrightarrow F^+$  is defined by the left-hand side of the formula (2.3) as well as  $\mathbf{L}^0$ . Further consider the space  $\Xi$  of 8-component vector functions defined at  $\Gamma$ . The components of vectors  $\xi \in \Xi$  contain the values of  $u, v, p, \rho$ , and their normal derivatives at  $\Gamma$ .  $\Xi$  is the space of clear traces of functions  $\mathbf{u}$  belonging to  $U^+$  [7], [8];  $\mathbf{Tr} : U^+ \longrightarrow \Xi$  is the clear trace operator. Note that this clear trace is presumably not the minimal one [8]. Generalized potential with the density belonging to  $\Xi$  is defined as follows:

(2.12) 
$$\mathbf{P}_{\overline{D}}\xi \stackrel{\text{def}}{=} \mathbf{u} - \mathbf{GL}\,\mathbf{u}, \ \mathbf{P}_{\overline{D}}: \ \Xi \longrightarrow U^+,$$

where  $\mathbf{u} \in U^+$  in (2.12) is such that  $\mathbf{Tr} \, \mathbf{u} = \xi$  and is arbitrary in the rest. It is shown in [7], [8] that the potential  $\mathbf{P}_{\overline{D}}\xi$  depends only on  $\xi$  but not on the choice of  $\mathbf{u} \in U^+$ ,  $\mathbf{Tr} \, \mathbf{u} = \xi$ , in formula (2.12).

Furthermore, let us introduce the operator

(2.13) 
$$\mathbf{P}_{\Gamma} \stackrel{\text{def}}{=} \mathbf{Tr} \, \mathbf{P}_{\overline{D}}, \ \mathbf{P}_{\Gamma} : \ \Xi \longrightarrow \Xi,$$

which is a projection, as shown in [7], [8]. The following statement, playing a fundamental role in all our constructions, holds for  $\mathbf{P}_{\Gamma}$  [7], [8]:  $\{\xi \in \Xi, \xi = \mathbf{P}_{\Gamma}\xi\} \Leftrightarrow$  $\{\exists \mathbf{u} \in U^+, \mathbf{L} \mathbf{u} = \mathbf{0}, \text{ and } \xi = \mathbf{Tr} \mathbf{u}\}$ . It means that the equation  $\mathbf{L} \mathbf{u} = \mathbf{0}$ , i.e., (2.3) being considered in D with conditions (2.5) and (2.9) at  $\partial D^0$ , is equivalent to the boundary equation

(2.14) 
$$\xi = \mathbf{P}_{\Gamma}\xi$$

Equation (2.14) contains the operator  $\mathbf{P}_{\Gamma}$  which is a generalization and modification of boundary projections introduced for the first time by Calderon [9] and Seeley [10] (see also [7], [8]). Due to the equivalence of (2.14) and (2.3), (2.5), (2.9) one can use (2.14) as an ABC at  $\Gamma$  for solving (2.1) in  $D_{\text{in}}$ . Indeed, (2.14) involves only the variables  $\xi \in \Xi$  determined at  $\Gamma$ . It is evident that these variables can be obtained using only the data from inside  $\overline{D}_{\text{in}}$ ; therefore (2.14) completes the problem in  $\overline{D}_{\text{in}}$ . The action of operator  $\mathbf{P}_{\Gamma}$  is spatially nonlocal. The specific procedure of numerical implementation of the nonlocal ABCs is described in §5. Note, in addition, that if (2.14) is valid then the solution of  $\mathbf{L} \mathbf{u} = \mathbf{0}$  in D with the clear trace  $\xi$  is unique and can be restored according to  $\xi$  found from (2.14) with the help of the generalized Green formula [7], [8],

$$\mathbf{u} = \mathbf{P}_{\overline{D}} \boldsymbol{\xi}.$$

Moreover, assuming that suitable norms are introduced in the spaces under consideration and that the original coupled problem (1.2) is well posed in these norms, it is possible to show [7], [8] that the boundary equation with projection (2.14) is also well posed.

3. Auxiliary problem with periodic boundary bonditions. By somewhat increasing the demands made of functions  $\mathbf{u}^0 \in U^0$  we will now show how to pass from AP (2.11) to the new AP with periodic boundary conditions in the *y*-direction, i.e., to the problem to be solved in some finite domain.

THEOREM 3.1. Let w(y) be defined on  $(-\infty, +\infty)$ , w(y), yw(y),  $y^2w(y) \in L_1(-\infty, +\infty)$ , w'(y),  $w''(y) \in C(-\infty, +\infty) \cap L_1(-\infty, +\infty)$ . Denote its Fourier transform as  $\hat{w}(\alpha)$ , so that

(3.1) 
$$w(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{w}(\alpha) e^{i\alpha y} d\alpha.$$

Let us associate with w(y) a periodic function  $w_Y(y)$  with the period Y. We determine this function by its Fourier series

(3.2) 
$$w_Y(y) = \sum_{k=-\infty}^{\infty} c_k e^{i k y \frac{2\pi}{Y}},$$
$$c_k = \frac{\sqrt{2\pi}}{Y} \hat{w}(\alpha_k),$$
$$\alpha_k = \frac{2\pi k}{Y}, \quad k = 0, \pm 1, \pm 2, \dots$$

Then, for any finite interval  $(-\breve{y}, \breve{y})$  and for any  $\epsilon > 0$  one may choose such a number  $Y_{\epsilon}$  that the estimate

۰.

$$|w(y) - w_Y(y)| < \epsilon, \quad y \in (-\breve{y}, \,\breve{y})$$

holds for any  $Y \geq Y_{\epsilon}$ .

Proof. Note that the Fourier transform  $\hat{w}(\alpha) = 1/\sqrt{2\pi} \int_{-\infty}^{\infty} w(y)e^{-i\alpha y} dy$  exists under the assumptions of Theorem 3.1. It has the second derivative  $\hat{w}''(\alpha)$ ; moreover,  $|\hat{w}(\alpha)| \leq c|\alpha|^{-2}, \alpha \longrightarrow \pm \infty$  [14]. By virtue of this estimate series (3.2) is majorized by the convergent number series:  $\sum_{k} |c_k| \leq \sqrt{2\pi}/Y \sum_{k} |\hat{w}(\alpha_k)| \leq \text{const}$  and therefore  $w_Y(y)$  (see (3.2)) is a continuous periodic function. Introduce a constant size mesh on the  $\alpha$ -axis:  $\alpha_k = k h_{\alpha}, \ k = 0, \pm 1, \pm 2, \ldots, \ h_{\alpha} = 2\pi/Y = \text{const}, \text{ and note that}$ one can represent the series (3.2) as follows:

(3.3) 
$$w_Y(y) = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \hat{w}(\alpha_k) e^{i\alpha_k y} h_{\alpha}.$$

Formula (3.3) may be obtained while replacing integral (3.1) by an approximate quadrature formula of rectangles.

Let us now estimate the value  $|w(y) - w_Y(y)|$ . Take some  $\mathcal{A} \equiv h_{\alpha}K > 0$  (arbitrary for the present):

$$\begin{split} |w(y) - w_Y(y)| &= \left| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{w}(\alpha) e^{i\alpha y} d\alpha - \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \hat{w}(\alpha_k) e^{i\alpha_k y} h_\alpha \right| \\ &\leq \frac{1}{\sqrt{2\pi}} \left| \int_{-\mathcal{A}}^{\mathcal{A}} \hat{w}(\alpha) e^{i\alpha y} d\alpha - \sum_{k=-K}^{K} \hat{w}(\alpha_k) e^{i\alpha_k y} h_\alpha \right| \\ &+ \frac{1}{\sqrt{2\pi}} \left| \int_{|\alpha| > \mathcal{A}} \hat{w}(\alpha) e^{i\alpha y} d\alpha - \sum_{|k| > K} \hat{w}(\alpha_k) e^{i\alpha_k y} h_\alpha \right| \end{split}$$

$$\stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}} |\cdot|_1 + \frac{1}{\sqrt{2\pi}} |\cdot|_2$$

Expression  $|\cdot|_1$  is an error of the quadrature formula of rectangles on the finite interval  $(-\mathcal{A}, \mathcal{A})$ . It is easy to see that

$$\frac{1}{\sqrt{2\pi}} |\cdot|_{1} \leq \operatorname{const} \cdot h_{\alpha}^{2} \mathcal{A} \max_{\alpha \in (-\mathcal{A}, \mathcal{A})} \left| \frac{d^{2}}{d\alpha^{2}} \left( \hat{w}(\alpha) e^{i\alpha y} \right) \right|$$
$$= \operatorname{const} \cdot h_{\alpha}^{2} \mathcal{A} \max_{\alpha \in (-\mathcal{A}, \mathcal{A})} \left| \left( \hat{w}''(\alpha) + 2iy \, \hat{w}'(\alpha) - y^{2} \, \hat{w}(\alpha) \right) e^{i\alpha y} \right|$$
$$\leq \operatorname{const} \cdot h_{\alpha}^{2} \mathcal{A} \max_{\alpha \in (-\mathcal{A}, \mathcal{A})} \left( |\hat{w}''(\alpha)| + 2|y| |\hat{w}'(\alpha)| + y^{2} |\hat{w}(\alpha)| \right)$$

$$= (c_1 + c_2|y| + c_3y^2) \mathcal{A}h_{\alpha}^2, \quad c_1, c_2, c_3 > 0.$$

For the second term we get

$$\frac{1}{\sqrt{2\pi}} |\cdot|_2 \le \frac{1}{\sqrt{2\pi}} \left( \int_{|\alpha| > \mathcal{A}} |\hat{w}(\alpha)| d\alpha + h_\alpha \sum_{|k| > K} |\hat{w}(\alpha_k)| \right)$$
$$\le \frac{2c}{\sqrt{2\pi}} \left( \int_{|\alpha| > \mathcal{A}} \alpha^{-2} d\alpha + h_\alpha \sum_{|k| > K} (k h_\alpha)^{-2} \right) \le \frac{c_4}{\mathcal{A}}, \quad c_4 > 0.$$

Thus,

(3.4) 
$$|w(y) - w_Y(y)| \le (c_1 + c_2|y| + c_3y^2) \mathcal{A}h_{\alpha}^2 + \frac{c_4}{\mathcal{A}}.$$

Now choose sufficiently large  $Y_{\epsilon}$  (or sufficiently small  $h_{\alpha_{\epsilon}} = 2\pi Y_{\epsilon}^{-1}$ ) so that the inequality

(3.5) 
$$c_0 \mathcal{A} h_\alpha^2 + \frac{c_4}{\mathcal{A}} < \epsilon,$$

where  $c_0 \stackrel{\text{def}}{=} \max_{y \in (-\check{y},\check{y})} (c_1 + c_2|y| + c_3y^2)$ ,  $c_0 > 0$ , would have a real positive solution  $\mathcal{A}$  of the specific type considered above; namely,  $\mathcal{A} = h_\alpha \cdot K$ , K is natural. In such a case, the coefficients of (3.5) should satisfy a somewhat stronger condition than simply the condition guaranteeing the existence of any real positive solution. For example, the solution  $\mathcal{A}$  possessing the above properties surely exists if one demands that the distance between the (real) roots of the corresponding quadratic equation (see (3.5)) is not less than  $h_\alpha$ . This results in the following inequality with respect to  $h_\alpha$ :

(3.6) 
$$\epsilon^2 - 4c_0c_4 h_\alpha^2 - c_0^2 h_\alpha^6 \ge 0.$$

All the real positive solutions of (3.6) fill the semi-interval  $(0, h_{\alpha_{\epsilon}}]$ , where  $h_{\alpha_{\epsilon}}$  is a positive root of the equation

$$\epsilon^2 - 4c_0c_4 h_\alpha^2 - c_0^2 h_\alpha^6 = 0,$$

which evidently always exists and is unique. It is clear that  $h_{\alpha_{\epsilon}} < \epsilon / (2\sqrt{c_0c_4})$  where the expression on the right-hand side of the last inequality yields the maximal value of mesh size  $h_{\alpha}$  which still guarantees the existence of real solution  $\mathcal{A}$  to (3.5). Now designate  $Y_{\epsilon} = 2\pi/h_{\alpha_{\epsilon}}$  and get that  $\forall Y \geq Y_{\epsilon}$  one may always find  $\mathcal{A}$  of the required type. Then, by virtue of (3.4) the estimate  $|w(y) - w_Y(y)| < \epsilon$  holds for any  $y \in$  $(-\breve{y}, \breve{y})$  which is the statement of the theorem.  $\Box$ 

Note that it is presumably also possible to get analogous results for the weaker assumptions about the smoothness of the function w(y) and about the rate of its decrease while  $y \longrightarrow \pm \infty$ , as well as for the stronger types of convergence (rather than simple uniform convergence).

However, the formulation of Theorem 3.1 as given above already provides the principle reason to pass from the AP (2.11) to the problem with periodic boundary conditions in the y-direction. Indeed, the boundary projection  $\mathbf{P}_{\Gamma}$  operates with the functions determined only at  $\Gamma$  and its difference analogue (see §5) operates with the functions determined "near"  $\Gamma$ . Therefore, we will need to know the solution of AP only in a small neighborhood of the curve  $\Gamma$ . If one demands from the rectangle  $(0, X) \times (-\breve{y}, \breve{y})$  to contain the necessary neighborhood then the solution of (2.11) and the solution of the periodic problem do not differ from one another within the chosen accuracy  $\epsilon$  on the set that is of interest to us. In doing so, the period Y of the function we use to approximate the original solution on  $(0, X) \times (-\breve{y}, \breve{y})$  increases while the permissible error  $\epsilon$  decreases.

Further we will consider the new periodic AP retaining all the old designations for the new domain and function spaces defined in it and supplying them with only the additional subscript "Y":

(3.7) 
$$\mathbf{L}^{0}\mathbf{u}_{Y}^{0} = \mathbf{f}_{Y}^{0}, \quad \mathbf{u}_{Y}^{0} \in U_{Y}^{0}, \quad \mathbf{f}_{Y}^{0} \in F_{Y}^{0},$$
$$D_{Y}^{0} = (0, X) \times \left(\frac{-Y}{2}, \frac{Y}{2}\right)$$

(we do not use a new subscript for the operator  $\mathbf{L}^0$  because its action is defined precisely in the same manner for both periodic functions and nonperiodic ones).

The spaces  $U_Y^0$  and  $F_Y^0$  consist of periodic functions:

(3.8) 
$$\mathbf{u}_{Y}^{0}(x, y) = \mathbf{u}_{Y}^{0}(x, y \pm Y),$$
$$\mathbf{f}_{Y}^{0}(x, y) = \mathbf{f}_{Y}^{0}(x, y \pm Y).$$

Concerning the right-hand sides, we are interested here in the functions with compact support belonging to some (small) neighborhood of the curve  $\Gamma$ ; consequently, supp  $\mathbf{f}_Y^0 \subset D_Y^0$ . Moreover, the conditions analogous to (2.9) are now included in the definition of  $U_Y^0$  for the discrete set of wavenumbers but no longer for the continuous one:

(3.9)  $\mathbf{S}_{Y}^{-}(k)\hat{\mathbf{v}}_{Y}^{0}(0,\,k) = \mathbf{0}, \quad k = 0,\,\pm 1,\,\pm 2,\ldots,$  $\mathbf{S}_{Y}^{+}(k)\hat{\mathbf{v}}_{Y}^{0}(X,\,k) = \mathbf{0}, \quad k = 0,\,\pm 1,\,\pm 2,\ldots,$ 

since by virtue of (3.8) we are going to use the following representation (compare with (3.2)):

(3.10) 
$$\mathbf{u}_{Y}^{0}(x, y) = \sum_{k=-\infty}^{\infty} \mathbf{c}_{k}(x) e^{i \, k y \frac{2\pi}{Y}},$$
where  $\mathbf{c}_{k}(x) = \frac{1}{Y} \int_{-\frac{Y}{2}}^{\frac{Y}{2}} \mathbf{u}_{Y}^{0}(x, y) e^{-i \, k y \frac{2\pi}{Y}} dy$ 

instead of (2.5).

The next stage is just the numerical solution of the AP (3.7)–(3.10), before which one ought to define the operators  $\mathbf{S}_{Y}^{-}(k)$  and  $\mathbf{S}_{Y}^{+}(k)$ . A detailed description of the numerical algorithm for the solution of (3.7)–(3.10) is contained in §4. Here we will dwell on the general question of convergence.

Consider some finite-difference schemes for (3.7)-(3.10):

(3.11) 
$$\mathbf{L}_{h}^{0}\mathbf{u}_{h,Y}^{0} = \mathbf{f}_{h,Y}^{0}, \quad \mathbf{u}_{h,Y}^{0} \in U_{h,Y}^{0}, \quad \mathbf{f}_{h,Y}^{0} \in F_{h,Y}^{0}.$$

The subscript "h" corresponds to the grid functions and operators acting in the spaces of grid functions. The Cartesian finite-difference grid in  $D_Y^0$  will be characterized by two sizes  $h_x$  and  $h_y$ ; therefore, one can treat h in (3.11) as a multi-index. The right-hand sides  $\mathbf{f}_{h,Y}^0 \in F_{h,Y}^0$  are determined, generally speaking, only on the set of "inner grid nodes." The structure of the latter depends on the stencil of the difference operator  $\mathbf{L}_h^0$  (see §§4 and 5). Let us now give the following definition.

DEFINITION 3.2. The solution of the difference problem (3.11) converges to the solution of the continuous problem (2.11) if  $\forall \breve{y} > 0$  such that  $(0, X) \times (-\breve{y}, \breve{y}) \supset D_{\text{in}}$  and  $\forall \epsilon > 0$  one can find a (sufficiently small) grid size  $h_{\epsilon}^{c} \equiv (h_{x_{\epsilon}}^{c}, h_{y_{\epsilon}}^{c})$  and a (sufficiently large) period  $Y_{\epsilon}^{c}$  such that  $\|\mathbf{u}_{h,Y}^{0} - [\mathbf{u}^{0}]_{h}\|_{\breve{y}} < \epsilon \quad \forall Y \ge Y_{\epsilon}^{c}, \forall h : h_{x} \le h_{x_{\epsilon}}^{c}, h_{y_{\epsilon}}$ .

Here  $[\mathbf{u}^0]_h$  designates the trace of the exact solution to (2.11) on the grid and  $\|\cdot\|_{\check{y}}$  is the *C*-norm in the space of grid vector functions (columns of height 4) determined on  $(0, X) \times (-\check{y}, \check{y})$ :  $\|\cdot\|_{\check{y}} = \max_{u, v, p, \rho} \max_{(x, y)_h \in (0, X) \times (-\check{y}, \check{y})} |\cdot|$ , where  $(x, y)_h$  are the grid nodes; i.e., the points of kind  $(mh_x, jh_y)$ , m, j are integers. Note that the type of convergence considered in Definition 3.2 (uniform convergence) is the same as that in Theorem 3.1. The norms used here and in Theorem 3.1 should be consistent with each other as will be seen from further considerations.

Definition 3.2 differs somewhat from the usual definition of convergence. First, we consider convergence not in the whole  $D^0$  but only on each specific subset of grid nodes from the rectangle  $(0, X) \times (-\breve{y}, \breve{y})$ . Second, the finite-dimensional spaces  $U_{h,Y}^0$  where an approximate solution is to be found are characterized not only by the grid size  $h \equiv (h_x, h_y)$  but also by the additional parameter Y. Actually we consider the convergence while  $(h_x, h_y, Y) \longrightarrow (0, 0, +\infty)$  and in so doing, the decrease in grid size and the increase in period are not independent.

Let us show qualitatively the dependence between period and size. One can estimate from above the value  $\|\mathbf{u}_{h,Y}^0 - [\mathbf{u}^0]_h\|_{\check{y}}$  (see Definition 3.2) as follows:

(3.12) 
$$\left\| \mathbf{u}_{h,Y}^{0} - [\mathbf{u}^{0}]_{h} \right\|_{\breve{y}} = \left\| \mathbf{u}_{h,Y}^{0} - [\mathbf{u}_{Y}^{0}]_{h} + [\mathbf{u}_{Y}^{0}]_{h} - [\mathbf{u}^{0}]_{h} \right\|_{\breve{y}}$$
$$\leq \left\| \mathbf{u}_{h,Y}^{0} - [\mathbf{u}_{Y}^{0}]_{h} \right\|_{\breve{y}} + \left\| [\mathbf{u}_{Y}^{0}]_{h} - [\mathbf{u}^{0}]_{h} \right\|_{\breve{y}}.$$

We require that the sum on the right-hand side of (3.12) be less than  $\epsilon$ ; this condition is evidently sufficient (but not necessary) for Definition 3.2 to be fulfilled. Assume that the scheme (3.11) possesses the convergence property in the "usual" sense, i.e., that the inequality

(3.13) 
$$\left\| \mathbf{u}_{h,Y}^{0} - \left[ \mathbf{u}_{Y}^{0} \right]_{h} \right\|_{Y} \le c_{Y} |h|^{q}, \quad |h| = |h_{x}| + |h_{y}|$$

holds in the fixed domain  $D_Y^0$  while the grid is being refined. Here, the exponent q > 0 determines the convergence rate. As to the value  $c_Y$  it is the constant which does not depend on the grid size but may depend, generally speaking, on the domain  $D_Y^0$  shape. Namely, in some cases  $c_Y$  may grow while the ratio Y/X increases under the fixed  $h_x$  and  $h_y$ . However, one may assume for the AP (3.7)–(3.10) that the values  $c_Y$  are totally bounded by some constant  $c_5$ . The numerical results (see §6) confirm this assumption experimentally.

Returning to inequality (3.12) we estimate from above the first term on its righthand side with the help of (3.13) and the second one on the basis of Theorem 3.1. (We consider, generally speaking, the continuous *C*-norm in Theorem 3.1 but it is evident that the required estimate will also remain valid for the traces of continuous solution on the grid.) The right-hand side of (3.12) should not exceed  $\epsilon > 0$ . Then, by virtue of (3.13), it is sufficient that the inequality

(3.14) 
$$\forall \epsilon > 0 \ \exists Y_{\epsilon}^c \text{ and } h_{\epsilon}^c : \ c_0 \mathcal{A} h_{\alpha}^2 + c_4 / \mathcal{A} + c_Y |h|^q < \epsilon$$

be valid. Now doing the same here as was done to justify the estimate (3.5) (see the proof of Theorem 3.1), let us drop  $\mathcal{A}$  from (3.14) considering all other variables as parameters. In other words, derive a sufficient condition providing that the inequality (3.14) has real positive solutions  $\mathcal{A}$  of the type  $\mathcal{A} = h_{\alpha} \cdot K$ ; K is natural:

(3.15) 
$$(\epsilon - c_Y |h|^q)^2 \ge 4c_0 c_4 \left(\frac{2\pi}{Y}\right)^2 + c_0^2 \left(\frac{2\pi}{Y}\right)^6.$$

The qualitative character of the relation between Y and h is seen from (3.15). Namely, to achieve the desirable accuracy  $\epsilon$  one ought to decrease the grid size h and to enlarge the period Y consistently.

Clearly, the formula (3.15) does not contain the definite means of choosing the AP parameters because the constants involved are not known in advance. However, this choice can be carried out experimentally. The results of some computations are presented in §6.

Here let us point out in addition that the best accuracy guaranteed by the above estimates is given by

(3.16) 
$$\epsilon_{\text{opt}} \equiv \epsilon_{\text{opt}}(h, Y) = \epsilon_{\text{opt}}(Y) + c_Y |h|^q$$

where  $\epsilon_{opt}(Y)$  is defined as follows. Let  $\mathcal{A}_0 = \arg \min_{\mathcal{A}} (c_0 \mathcal{A} h_{\alpha}^2 + c_4 / \mathcal{A}) = \sqrt{c_4/c_0}(1/h_{\alpha}) = \sqrt{c_4/c_0}(Y/2\pi)$ . Designate  $K_1 = [\mathcal{A}_0/h_{\alpha}] = [\mathcal{A}_0Y/2\pi]$ ,  $K_2 = K_1 + 1$ , where [·] represents the integer part. Then

(3.17) 
$$\epsilon_{\text{opt}}(Y) \stackrel{\text{def}}{=} \min\left(c_0 K_1 h_{\alpha}^3 + \frac{c_4}{K_1 h_{\alpha}}, c_0 K_2 h_{\alpha}^3 + \frac{c_4}{K_2 h_{\alpha}}\right)$$
$$= \min\left(c_0 K_1 \left(\frac{2\pi}{Y}\right)^3 + \frac{c_4}{K_1} \frac{Y}{2\pi}, c_0 K_2 \left(\frac{2\pi}{Y}\right)^3 + \frac{c_4}{K_2} \frac{Y}{2\pi}\right).$$

Evidently,  $\epsilon_{opt}(Y) \longrightarrow 0$  as  $Y \longrightarrow +\infty$ ; therefore, the case of  $c_Y = c_5 = \text{const} \epsilon_{opt}(h, Y)$  decreases while Y grows:  $\epsilon_{opt}(h, Y) \longrightarrow c_5|h|^q$  as  $Y \longrightarrow +\infty$ . Our construction of AP provides the realization of just this case in practical computations (§6). Moreover, if  $h \longrightarrow 0$  and Y is fixed then  $\epsilon_{opt}(h, Y)$  decreases being bounded from below by the positive value  $4\pi\sqrt{c_0c_4}/Y$ . This behavior corresponds to the "usual" convergence of the difference periodic solution to the continuous periodic one.

4. Numerical solution of the periodic auxiliary problem. We first specify the finite-difference scheme (3.11). Remember that the Cartesian grid with constant size in both directions

(4.1) 
$$\mathcal{N}^{0} = \{ (x_{m}, y_{j}) \equiv (mh_{x}, jh_{y} - Y/2) \mid h_{x}, h_{y} > 0, m = 0, \dots, M, M = X/h_{x}, j = 0, \dots, 2J + 1, 2J + 1 = Y/h_{y} \}$$

is introduced in  $D_Y^0$ . We construct the second-order approximation to (3.7) on the grid  $\mathcal{N}^0$  (the operator  $\mathbf{L}^0$  from (3.7) is defined by the left-hand side of (2.3)):

$$\begin{aligned} (4.2) \qquad \frac{\rho_{m+1,j}^{0} - \rho_{m-1,j}^{0}}{2h_{x}} + \frac{u_{m+1,j}^{0} - u_{m-1,j}^{0}}{2h_{x}} + \frac{v_{m,j+1}^{0} - v_{m,j-1}^{0}}{2h_{y}} &= f_{1_{m,j}}^{0}, \\ \\ \frac{u_{m+1,j}^{0} - u_{m-1,j}^{0}}{2h_{x}} + \frac{p_{m+1,j}^{0} - p_{m-1,j}^{0}}{2h_{x}} - \frac{1}{Re} \left[ \frac{4}{3} \frac{u_{m+1,j}^{0} - 2u_{m,j}^{0} + u_{m-1,j}^{0}}{h_{x}^{2}} \right] \\ + \frac{v_{m+1,j+1}^{0} - v_{m+1,j-1}^{0} - v_{m-1,j+1}^{0} + v_{m-1,j-1}^{0}}{12h_{x}h_{y}} + \frac{u_{m,j+1}^{0} - 2u_{m,j}^{0} + u_{m,j-1}^{0}}{h_{y}^{2}} \right] \\ = f_{2_{m,j}}^{0}, \\ \\ \frac{v_{m+1,j}^{0} - v_{m-1,j}^{0}}{2h_{x}} + \frac{p_{m,j+1}^{0} - p_{m,j-1}^{0}}{2h_{y}} - \frac{1}{Re} \left[ \frac{v_{m+1,j}^{0} - 2v_{m,j}^{0} + v_{m-1,j}^{0}}{h_{x}^{2}} \right] \\ + \frac{u_{m+1,j+1}^{0} - u_{m+1,j-1}^{0} - u_{m-1,j+1}^{0} + u_{m-1,j-1}^{0}}{12h_{x}h_{y}} - \frac{1}{Re} \left[ \frac{v_{m+1,j}^{0} - 2v_{m,j}^{0} + v_{m,j-1}^{0}}{h_{x}^{2}} \right] \\ = f_{3_{m,j}}^{0}, \\ \\ \frac{p_{m+1,j}^{0} - p_{m-1,j}^{0}}{2h_{x}} - \frac{1}{(Ma)^{2}} \frac{\rho_{m+1,j}^{0} - \rho_{m-1,j}^{0}}{h_{x}^{2}} + \frac{p_{m,j+1}^{0} - 2p_{m,j}^{0} + p_{m,j-1}^{0}}{h_{y}^{2}} \\ - \frac{1}{\gamma(Ma)^{2}} \left( \frac{\rho_{m+1,j}^{0} - 2\rho_{m,j}^{0} + \rho_{m-1,j}^{0}}{h_{x}^{2}} + \frac{\rho_{m,j+1}^{0} - 2\rho_{m,j}^{0} + \rho_{m,j-1}^{0}}{h_{y}^{2}} \right) \right] = f_{4_{m,j}}^{0}, \\ \\ m = 1, 2, \dots, M - 1, \quad j = 0, 1, \dots, 2J. \end{aligned}$$

We will designate  $\mathbf{u}_{m,j}^{0} = (u_{m,j}^{0}, v_{m,j}^{0}, p_{m,j}^{0}, \rho_{m,j}^{0})^{T}$ ,  $\mathbf{f}_{m,j}^{0} = (f_{1_{m,j}}^{0}, f_{2_{m,j}}^{0}, f_{3_{m,j}}^{0}, f_{4_{m,j}}^{0})^{T}$  as before. The difference scheme (4.2) is written using the 9-node stencil  $3 \times 3$ ;

therefore, if the solution is determined on the whole grid  $\mathcal{N}^0$  from (4.1) then the righthand sides are determined only in the inner nodes  $m = 1, \ldots, M - 1$  (see page 1367). Periodic boundary conditions in the *y*-direction are now represented as follows:

(4.3) 
$$\mathbf{u}_{m,0}^{0} = \mathbf{u}_{m,2J+1}^{0}, \quad m = 0, \dots, M,$$
$$\mathbf{u}_{m,-1}^{0} = \mathbf{u}_{m,2J}^{0}, \quad m = 0, \dots, M.$$

We wish to find a solution for (4.2)-(4.3) in the form of the finite Fourier series, before which we have to construct the boundary conditions of type (3.9) for m = 0and m = M. Substituting the expressions

(4.4) 
$$\mathbf{u}_{m,j}^{0} = \sum_{k=-J}^{J} \hat{\mathbf{u}}_{m,k}^{0} e^{i\,kjh_{y}\frac{2\pi}{Y}}, \quad m = 0, \dots, M, \ j = 0, \dots, 2J,$$
$$\mathbf{f}_{m,j}^{0} = \sum_{k=-J}^{J} \hat{\mathbf{f}}_{m,k}^{0} e^{i\,kjh_{y}\frac{2\pi}{Y}}, \quad m = 1, \dots, M-1, \ j = 0, \dots, 2J$$

into (4.2) where the coefficients of the series (4.4) are defined by

(4.5) 
$$\hat{\mathbf{u}}_{m,k}^{0} = \frac{1}{2J+1} \sum_{j=0}^{2J} \mathbf{u}_{m,j}^{0} e^{-i \, k j h_{y} \frac{2\pi}{Y}}, \quad m = 0, \dots, M, \quad k = -J, \dots, J,$$
$$\hat{\mathbf{f}}_{m,k}^{0} = \frac{1}{2J+1} \sum_{j=0}^{2J} \mathbf{f}_{m,j}^{0} e^{-i \, k j h_{y} \frac{2\pi}{Y}}, \quad m = 1, \dots, M-1, \quad k = -J, \dots, J,$$

we obtain for each wavenumber  $k = -J, \ldots, J$  the following second-order system of ordinary difference equations.

(4.6) 
$$\frac{\hat{\rho}_{m+1,k}^0 - \hat{\rho}_{m-1,k}^0}{2h_x} + \frac{\hat{u}_{m+1,k}^0 - \hat{u}_{m-1,k}^0}{2h_x} + s_k \, \hat{v}_{m,k}^0 = \hat{f}_{1_{m,k}}^0,$$

$$\begin{aligned} \frac{\hat{u}_{m+1,k}^{0} - \hat{u}_{m-1,k}^{0}}{2h_{x}} + \frac{\hat{p}_{m+1,k}^{0} - \hat{p}_{m-1,k}^{0}}{2h_{x}} - \frac{1}{Re} \left[ \frac{4}{3} \frac{\hat{u}_{m+1,k}^{0} - 2\hat{u}_{m,k}^{0} + \hat{u}_{m-1,k}^{0}}{h_{x}^{2}} \right. \\ \\ \left. + \frac{1}{3} s_{k} \frac{\hat{v}_{m+1,k}^{0} - \hat{v}_{m-1,k}^{0}}{2h_{x}} - t_{k} \hat{u}_{m,k}^{0} \right] = \hat{f}_{2m,k}^{0}, \\ \\ \frac{\hat{v}_{m+1,k}^{0} - \hat{v}_{m-1,k}^{0}}{2h_{x}} + s_{k} \hat{p}_{m,k}^{0} - \frac{1}{Re} \left[ \frac{\hat{v}_{m+1,k}^{0} - 2\hat{v}_{m,k}^{0} + \hat{v}_{m-1,k}^{0}}{h_{x}^{2}} \right] \\ \\ \left. + \frac{1}{3} s_{k} \frac{\hat{u}_{m+1,k}^{0} - \hat{u}_{m-1,k}^{0}}{2h_{x}} - \frac{1}{3} t_{k} \hat{v}_{m,k}^{0} \right] = \hat{f}_{3m,k}^{0}, \\ \\ \frac{\hat{p}_{m+1,k}^{0} - \hat{p}_{m-1,k}^{0}}{2h_{x}} - \frac{1}{(Ma)^{2}} \frac{\hat{\rho}_{m+1,k}^{0} - \hat{\rho}_{m-1,k}^{0}}{2h_{x}} - \frac{\gamma}{Re} Pr} \left[ \frac{\hat{p}_{m+1,k}^{0} - 2\hat{p}_{m,k}^{0} + \hat{p}_{m-1,k}^{0}}{h_{x}^{2}} \right] \end{aligned}$$

$$-t_k \, \hat{p}^0_{m,k} - \frac{1}{\gamma(Ma)^2} \left( \frac{\hat{\rho}^0_{m+1,k} - 2\hat{\rho}^0_{m,k} + \hat{\rho}^0_{m-1,k}}{h_x^2} - t_k \, \hat{\rho}^0_{m,k} \right) \right] = \hat{f}^0_{4_{m,k}},$$

$$m = 1, \dots, M - 1, \ k = -J, \dots, J, \ s_k = \frac{i \sin k \tilde{h}_y}{h_y}, \ t_k = \frac{4}{h_y^2} \sin^2 \frac{k \tilde{h}_y}{2}, \ \tilde{h}_y = h_y \frac{2\pi}{Y}.$$

Introducing the additional variables (compare with (2.7))

(4.7)  

$$\hat{u}_{m,k}^{0} - \hat{u}_{m-1,k}^{0} - h_{x} \chi_{m-1,k} = 0, \\
\hat{v}_{m,k}^{0} - \hat{v}_{m-1,k}^{0} - h_{x} \eta_{m-1,k} = 0, \\
\hat{p}_{m,k}^{0} - \hat{p}_{m-1,k}^{0} - h_{x} \zeta_{m-1,k} = 0, \\
\hat{\rho}_{m,k}^{0} - \hat{\rho}_{m-1,k}^{0} - h_{x} \psi_{m-1,k} = 0, \\
m = 1, \dots, M,$$

and designating

$$\begin{aligned} \hat{\mathbf{v}}_{m,k}^{0} &= \left(\chi, \, \eta, \, \zeta, \, \psi, \, \hat{u}^{0}, \, \hat{v}^{0}, \, \hat{p}^{0}, \, \hat{\rho}^{0}\right)^{T} \Big|_{m,k}, \\ \hat{\mathbf{g}}_{m,k}^{0} &= \left(\hat{f}_{1}^{0}, \, h_{x}\hat{f}_{2}^{0}, \, h_{x}\hat{f}_{3}^{0}, \, h_{x}\hat{f}_{4}^{0}, \, 0, \, 0, \, 0, \, 0\right)^{T} \Big|_{m,k}, \end{aligned}$$

we rewrite (4.6) as the system of eight first-order ordinary difference equations:

(4.8) 
$$\mathbf{A}_k \hat{\mathbf{v}}_{m,k}^0 + \mathbf{B}_k \hat{\mathbf{v}}_{m-1,k}^0 = \hat{\mathbf{g}}_{m,k}^0,$$
$$m = 1, \dots, M, \ k = -J, \dots, J,$$

where the matrices  $\mathbf{A}_k$  and  $\mathbf{B}_k$  are given by the following expressions:

Note that the first-order system of ODEs (2.8) and the corresponding first-order discrete system (4.8) are of different dimensionalities (7 and 8, respectively) due to the reasons of simplicity. We think that the easiest way to obtain a first-order system on a discrete level for the specific scheme (4.2), (4.6) is simply to introduce four additional variables, whereas in (2.7) we introduced only three. While deriving the boundary conditions for (4.8) (see the formulae (4.13), (4.14)), the dimensionality growth may cause an additional condition (for each wavenumber k) to appear. However, the discrete boundary conditions at x = 0 and at x = X will be based on the same idea as in §2 (see (2.9), (2.10)). Namely, we prohibit all the nondecreasing solutions on both ends of the interval, and thus, the addition of an extra variable ( $\psi$  in (4.8)) should not change the far-field behavior of hydrodynamic variables  $u, v, p, \rho$ . Therefore, this additional condition should present no contradiction to (2.9) and should not disturb convergence of the discrete solution  $\mathbf{u}_{m,j}^0$  to the continuous solution  $\mathbf{u}^0(x, y)$ (in the sense of Definition 3.2). Regretfully, the question of "identifying" this specific additional condition in the nonlocal matrix relation (see (4.13)) and of "establishing the one-to-one correspondence" between the continuous conditions (2.9) and those discrete conditions which do approximate (2.9) (i.e., which are not additional) seems to be rather difficult. However, our numerical experiments actually provide us with sufficient reasons to use just the system (4.8) of eight equations and the corresponding conditions (4.13); see §6.

Now we proceed directly to the construction of boundary conditions for the difference AP. System (4.8) should be supplied for each  $k = -J, \ldots, J$  with boundary conditions at m = 0 and m = M. To do this let us formally consider the extension of (4.8) on the infinite (in the x-direction) grid

(4.11) 
$$\mathbf{A}_{k}\hat{\mathbf{v}}_{m,k}^{0} + \mathbf{B}_{k}\hat{\mathbf{v}}_{m-1,k}^{0} = \hat{\mathbf{g}}_{m,k}^{0},$$
$$m = 0, \pm 1, \pm 2, \dots, \ k = -J, \dots, J.$$

with  $\hat{\mathbf{g}}_{m,k}^0 = \mathbf{0}$  for  $m \leq 0$  and for  $m \geq M$ ; i.e., the system (4.11) is homogeneous outside the finite interval. We need to find a solution to (4.8) for  $m = 0, \ldots, M$  that would be a fragment of a solution to (4.11) which is bounded for all  $m = 0, \pm 1, \pm 2, \ldots$ . In reality the solution of (4.11) has to vanish while  $m \longrightarrow \pm \infty$  (see page 1361) for all  $k = -J, \ldots, J$ . However it turns out that it is not always possible. Actually, the behavior of the solutions to (4.11) in the "homogeneous" part of the domain is determined by the spectrum of the matrix

$$\mathbf{Q}_k = \mathbf{A}_k^{-1} \mathbf{B}_k.$$

If it occurs for all the eigenvalues  $\mu_s(k)$ ,  $s = 1, \ldots, 8$  of  $\mathbf{Q}_k$  that  $|\mu_s(k)| \neq 1$ , then the solution of (4.11) vanishing at infinity  $\|\hat{\mathbf{v}}_{m,k}^0\| \longrightarrow 0$ ,  $m \longrightarrow \pm \infty$ , does exist for the given wavenumber k. This is evidently the case in the general situation. But it is also possible that there are some eigenvalues equal to 1 in absolute value:  $|\mu_s(k)| = 1$ . This means that the homogeneous system (4.11) admits, generally speaking, constant, oscillating, or polynomial solutions. In this case one can ensure that the solution of (4.11) will vanish only along one particular direction:  $m \longrightarrow -\infty$  or  $m \longrightarrow +\infty$ .

The directions  $m \longrightarrow -\infty$  and  $m \longrightarrow +\infty$  do not have the same rights in the problem under investigation. Namely, the former corresponds to the upstream propagation of perturbations (inflow) and the latter to the downstream propagation (outflow). Therefore, we will primarily demand that the solution to (4.11) vanishes while  $m \longrightarrow -\infty$ , i.e., along the upstream direction for all  $k = -J, \ldots, J$ . Regarding the downstream conditions  $(m \longrightarrow +\infty)$ , they depend on the structure of the  $\mathbf{Q}_k$  spectrum. We require here either a strict decrease or only boundedness of the solution for  $m \longrightarrow +\infty$  if the eigenvalue(s)  $|\mu_s(k)| = 1$  exist(s). Moreover, it is necessary to add the following important remark.

*Remark.* We assume that even when the eigenvalue  $\mu_s(k)$ :  $|\mu_s(k)| = 1$  is multiple, the matrix  $\mathbf{Q}_k$  still has as many linearly independent eigenvectors corresponding to this eigenvalue  $\mu_s(k)$  as its multiplicity. This means that we do not consider polynomially growing solutions. For the small matrices  $\mathbf{Q}_k$  (8 × 8) one can easily verify numerically that Jordan blocks of order more than 1 are absent in the canonical form of  $\mathbf{Q}_k$  (see §6).

Thus, the boundary conditions of the type (3.9) for system (4.8) should prohibit at m = 0 all solutions which are nondecreasing to the left. They should also prohibit at m = M all solutions which are increasing to the right. It is easy to see that one can write such conditions as follows:

(4.13) 
$$\mathbf{S}_{h,Y}^{-}(k)\hat{\mathbf{v}}_{0,k}^{0} = \mathbf{0}, \quad k = 0, \pm 1, \pm 2, \dots \pm J,$$
$$\mathbf{S}_{h,Y}^{+}(k)\hat{\mathbf{v}}_{M,k}^{0} = \mathbf{0}, \quad k = 0, \pm 1, \pm 2, \dots \pm J$$

where  
(4.14) 
$$\mathbf{S}_{h,Y}^{-}(k) = \prod_{|\mu_{s}(k)| > 1} (\mathbf{Q}_{k} - \mu_{s}(k)\mathbf{I})$$
$$\mathbf{S}_{h,Y}^{+}(k) = \prod_{|\mu_{s}(k)| \le 1} (\mathbf{Q}_{k} - \mu_{s}(k)\mathbf{I})$$

I in (4.14) is an identity matrix of the eighth order and the matrix products are computed according to the eigenvalues multiplicities. The eigenvalues  $\mu_s(k)$ ,  $s = 1, \ldots, 8$ ,  $k = -J, \ldots, J$  were calculated numerically with the help of standard NAG routines while the computations were carried out. The accuracy of such calculations for the matrices of low order (8 × 8) is extremely high. Boundary conditions (4.13) actually select the solutions of (4.8) according to the character of their growth for any matrix  $\mathbf{Q}_k$ . In particular,  $\mathbf{Q}_k$  may have no basis consisting of eigenvectors; i.e., there may be Jordan blocks of order more than 1 in the canonical form of  $\mathbf{Q}_k$ . A rigorous proof of this statement is contained in the Appendix (§7).

Thus, it is now possible to give the final formulation of the difference AP: to find a solution to the system (4.2) in  $D_Y^0$  on the grid  $\mathcal{N}^0$  (see (4.1)) with periodic conditions (4.3) in the y-direction and boundary conditions (4.13) at x = 0 and x = X. Conditions (4.13) are imposed on the Fourier components (4.5), (4.7) of the difference solution and the matrices  $\mathbf{S}_{h,Y}^-(k)$ ,  $\mathbf{S}_{h,Y}^+(k)$  involved are defined by means of the relations (4.14), (4.12), (4.9), (4.10).

"Disagreement" still exists in this case. We recall that conditions (4.13) constructed above permit in certain situations the solutions which are nondecreasing to the right. On the other hand, one should be able to make the solution of the coupled problem (1.2) asymptotically more precise by means of the size of  $D_{\rm in}$  enlargement (see §1). This requirement means that the solution of the linearized problem has to vanish while  $|x| \longrightarrow \infty$ . In the case of  $|\mu_s(k)| \neq 1$  both a decrease and an increase of the solutions under investigation have an exponential character (see Appendix (§7)). It is, therefore, easy to check the validity of (1.5) directly. However, the case  $|\mu_s(k)| = 1$ requires separate consideration.

We do not rigorously justify here the possibility of using conditions (4.13) because it presents significant difficulties to obtain analytical expressions for eigenvalues of  $\mathbf{Q}_k$ . However, the computations performed for different flow regimes and for different grids show that there are no eigenvalues with unit module for all  $k \neq 0$ ,  $\forall k =$  $\pm 1, \pm 2, \ldots, \pm J$  :  $|\mu_s(k)| \neq 1$ ,  $s = 1, \ldots, 8$ . And only for k = 0 the (multiple) eigenvalue  $|\mu(0)| = 1$  exists. Having no opportunity to verify it experimentally we assume that this property takes place not only for the finite Fourier series (4.4) but also for the infinite one (3.10); i.e., the eigenvalue with unit module may appear only for k = 0. Proceeding from the Fourier series (3.10) to the integral (2.5) or similarly from (3.3) to (3.1) corresponds formally to the "diminution of the weight of each coefficient  $\mathbf{c}_k$ " to zero since  $\mathbf{c}_k \sim Y^{-1} \sim h_{\alpha}$ , where  $Y \longrightarrow +\infty \Leftrightarrow h_{\alpha} \longrightarrow +0$  (see (3.2)). Therefore, if the matrices  $\mathbf{Q}(\alpha)$ ,  $\alpha \in (-\infty, +\infty)$  from (2.8) are such that  $\Re \lambda_s(\alpha) = 0$  may occur only for  $\alpha = 0$ , then one has reason to believe that in reality the solution of the continuous AP will nevertheless vanish while  $x \rightarrow +\infty$  although the conditions (2.9) formally guarantee only its boundedness in such a case.

Note that  $\Re \lambda_s(0) = 0$  corresponds to a constant or oscillating solution of (2.8) just as  $|\mu_s(0)| = 1$  corresponds to constant or oscillating solutions of (4.8). We do not consider here the polynomial growth since we suppose that the eigenvectors of  $\mathbf{Q}(0)$  always form a complete linearly independent system (compare with the remark on page 1373). The following lemma shows the behavior of the solution to AP for  $x \longrightarrow +\infty$ .

LEMMA 4.1. Let  $\hat{w}(X, \alpha)$  be a bounded absolutely integrable function while  $\alpha \in (-\infty, +\infty)$  and let  $\hat{w}(x, \alpha) = \hat{w}(X, \alpha)e^{-\lambda(\alpha)x}$  for x > X,  $\lambda(\alpha)$  is a continuous function,  $\Re\lambda(0) = 0$ ,  $\forall \delta > 0$ :  $\min_{|\alpha| \ge \delta} \Re\lambda(\alpha) > 0$ . Then  $\forall \epsilon > 0 \exists \breve{x} = \breve{x}(\epsilon)$  such that for all  $x \ge \breve{x}$  and for all  $y \in (-\infty, +\infty)$  the estimate  $|w(x, y)| < \epsilon$  holds, where  $w(x, y) = 1/\sqrt{2\pi} \int_{-\infty}^{\infty} \hat{w}(x, \alpha)e^{i\alpha y}d\alpha$ .

The condition of Lemma 4.1 corresponds exactly to the definition of the operator  $\mathbf{S}^+(\alpha)$  from (2.9). Indeed, this operator prohibits the solutions which grow while  $x \longrightarrow +\infty$ . Moreover, the function  $\hat{w}(x, \alpha)$  decreases exponentially while  $x \longrightarrow +\infty$  for all  $\alpha$  except  $\alpha = 0$  which corresponds to the remark on page 1373.

*Proof.* Let  $\delta > 0$ .

$$w(x, y) = \frac{1}{\sqrt{2\pi}} \left[ \int_{-\infty}^{-\delta} \hat{w}(x, \alpha) e^{i\alpha y} d\alpha + \int_{-\delta}^{\delta} \hat{w}(x, \alpha) e^{i\alpha y} d\alpha + \int_{\delta}^{+\infty} \hat{w}(x, \alpha) e^{i\alpha y} d\alpha \right]$$
$$\Rightarrow |w(x, y)| \le \left| \int_{-\infty}^{-\delta} \cdot + \int_{\delta}^{+\infty} \cdot \right| + \left| \int_{-\delta}^{\delta} \cdot \right|.$$

We estimate separately the integrals over finite and infinite intervals. By virtue of the conditions imposed on  $\hat{w}(X, \alpha)$ , inequalities

$$\left| \int_{-\delta}^{\delta} \cdot \right| \le b_1 \delta, \qquad \left| \int_{-\infty}^{-\delta} \cdot + \int_{\delta}^{+\infty} \cdot \right| \le b_2 e^{-\lambda_{\min} x}$$

hold, where  $b_1 = \text{const} > 0$ ,  $b_2 = \text{const} > 0$ ,  $\lambda_{\min} \equiv \lambda_{\min}^{\delta} = \lambda \left( \alpha_{\min}^{\delta} \right)$ , and  $\alpha_{\min}^{\delta} = \arg\min_{|\alpha| \ge \delta} \Re \lambda(\alpha)$ . Evidently,  $\Re \lambda_{\min} > 0$ . Now choose  $\delta = \epsilon / (2b_1)$  and thereafter  $\breve{x}(\epsilon) = -(\ln(\epsilon/2b_2)/\lambda_{\min})$ . Then we obtain:  $|w(x, y)| \le \epsilon$  for all  $(x, y) \in (\breve{x}, +\infty) \times (-\infty, +\infty)$ .

Note that the functions w,  $\hat{w}$  in Lemma 4.1 are scalar, but generalization to the vector case is evident.

Thus, although the solution of the difference AP obtained for each specific computation variant can really contain components nondecreasing along the downstream direction, the nearer it is to the true continuous solution the smaller is the relative contribution of these components; i.e., in the case of convergence (see Definition 3.2) the limitary function decreases while  $x \longrightarrow \pm \infty$ .

Finally, the questions of well-posedness for the general systems of ordinary difference equations are studied in detail in [15]. We do not repeat the corresponding analysis here; notice, however, that by virtue of the  $\mathbf{S}_{h,Y}^{-}(k)$  and  $\mathbf{S}_{h,Y}^{+}(k)$  definitions (see (4.14)), the following equality rank  $\mathbf{S}_{h,Y}^{-}(k)$  + rank  $\mathbf{S}_{h,Y}^{+}(k) = 8$  always holds (see Appendix (§7)); i.e., boundary conditions (4.13) are noncontradictory. We also note (see [15]) that for all  $k = -J, \ldots, J$  problem (4.8), (4.13) is uniquely solvable  $\forall \hat{\mathbf{g}}_{m,k}^{0}$ and well conditioned. A convenient technique for solving (4.8), (4.13) is the Godunov algorithm of orthogonal successive substitution [16]. The method [16] is suitable for integration of stiff systems including the case of variable coefficients. In our case we have modified and simplified the original algorithm using the fact that the coefficients of (4.8) are constant.

After the computation of the solution to (4.8) for each wavenumber  $k = -J, \ldots, J$ one obtains the solution of the whole difference AP on the grid  $\mathcal{N}^0$  (see (4.1)) by means of the inverse Fourier transform (4.4). Therefore, the Green operator  $\mathbf{G}_{h,Y}^0: F_{h,Y}^0 \longrightarrow$  $U_{h,Y}^0$  of the difference AP is defined. We will need this operator for constructing the difference ABCs.

5. Difference ABCs. We describe in the current section the constructions of the DPM (see [8] for details) as applied to the setting of ABCs at the boundary  $\Gamma$ . We emphasize here that the ABCs will be developed below directly for the finite-difference formulation of the problem. The DPM [8] provides us with such an opportunity, which is most convenient for practical computations.

The space  $U_{h,Y}^0$  of the solutions to the difference AP is determined on the grid  $\mathcal{N}^0$  from (4.1) and the space of its right-hand sides  $F_{h,Y}^0$  on the grid  $\mathcal{M}^0 \stackrel{\text{def}}{=} \mathcal{N}^0 \setminus \{(x_m, y_j) \mid m = 0, m = M, j = 0, \dots, 2J + 1\}$ .  $F_{h,Y}^0$  contains all the grid functions determined on  $\mathcal{M}^0$ . Designate  $D_Y = D_Y^0 \cap D_{\text{ex}}$  and define the grid sets:

$$\mathcal{M} = \left\{ (x_m, y_j) \mid (x_m, y_j) \in \mathcal{M}^0 \cap \overline{D}_Y \right\},$$
$$\mathcal{M}_{in} = \left\{ (x_m, y_j) \mid (x_m, y_j) \in \mathcal{M}^0 \cap D_{in} \right\}.$$

Let  $St_{m,j}$  be the stencil of the difference operator  $\mathbf{L}_h^0$  (see formula (4.2)), with its center in the node  $(x_m, y_j)$ ; i.e.,

$$St_{m,j} = \bigcup_{\substack{m'=m-1, m, m+1 \\ j'=j-1, j, j+1}} (x_{m'}, y_{j'}).$$

Then

$$\mathcal{N} \stackrel{\text{def}}{=} \bigcup_{(x_m, y_j) \in \mathcal{M}} St_{m, j}; \quad \mathcal{N}_{\text{in}} \stackrel{\text{def}}{=} \bigcup_{(x_m, y_j) \in \mathcal{M}_{\text{in}}} St_{m, j}; \quad \gamma \stackrel{\text{def}}{=} \mathcal{N} \cap \mathcal{N}_{\text{in}}$$

We will call the set  $\gamma$  the grid boundary (by analogy with the continuous boundary  $\Gamma$ ). Evidently,  $\gamma$  consists of those nodes of the grid  $\mathcal{N}^0$  which are located "not far from"  $\Gamma$ . Thereafter introduce the operators:  $\Theta_{\mathcal{M}}$  — restriction of the domain of grid function from  $\mathcal{M}^0$  to  $\mathcal{M}, \Theta^0_{\mathcal{M}}$  — complement of the grid function determined on  $\mathcal{M}$  by zero on  $\mathcal{M}^0 \backslash \mathcal{M} = \mathcal{M}_{in}$ , as well as the operators  $\Theta_{\mathcal{N}}$  and  $\Theta^0_{\mathcal{N}}$  acting analogously for the sets  $\mathcal{N}$  and  $\mathcal{N}^0$ , respectively. Define the following spaces of grid functions:

(5.1) 
$$U_{h,Y} = \left\{ \Theta_{\mathcal{N}} \mathbf{u}_{h,Y}^{0} \mid \mathbf{u}_{h,Y}^{0} \in U_{h,Y}^{0} \right\},$$
$$F_{h,Y} = \left\{ \Theta_{\mathcal{M}} \mathbf{f}_{h,Y}^{0} \mid \mathbf{f}_{h,Y}^{0} \in F_{h,Y}^{0} \right\}.$$

 $F_{h,Y}$  and  $U_{h,Y}$  from (5.1) are the analogues of  $F^+$  and  $U^+$  from §2. Here we do not introduce the spaces with superscript "+" since by virtue of the  $F_{h,Y}^0$  definition  $\forall \mathbf{f}_{h,Y}^0 \in F_{h,Y}^0$  :  $\Theta_{\mathcal{M}^0}(\mathcal{M})\mathbf{f}_{h,Y}^0 \in F_{h,Y}^0$ . We will define the action of the difference operator  $\mathbf{L}_h : U_{h,Y} \longrightarrow F_{h,Y}$  with the help of (4.2) in the same manner as the action of  $\mathbf{L}_{h}^{0}$ . Moreover, we introduce the operator  $\mathbf{G}_{h,Y}$ :  $F_{h,Y} \longrightarrow U_{h,Y}$ ;  $\forall \mathbf{f}_{h,Y} \in F_{h,Y}$ we define it as follows:  $\mathbf{G}_{h,Y}\mathbf{f}_{h,Y} \stackrel{\text{def}}{=} \Theta_{\mathcal{N}}\mathbf{G}_{h,Y}^{0}\Theta_{\mathcal{M}}^{0}\mathbf{f}_{h,Y}$  (compare with §2, page 1363). The space of difference clear traces  $\Xi_{h}$  [8] consists of all the 4-component vector functions  $\xi_{h}$  defined on the grid boundary  $\gamma$ , and the difference clear trace operator [8]  $\mathbf{Tr}_{h}$ :  $U_{h,Y} \longrightarrow \Xi_{h}$  simply truncates the domain of the corresponding function from  $\mathcal{N}$  to  $\gamma$ . Difference potential [8] (see also [7]) with the density from the space  $\Xi_{h}$  of clear traces is defined by the formula

(5.2) 
$$\mathbf{P}_{\mathcal{N}}\xi_{h} = \mathbf{u}_{h,Y} - \mathbf{G}_{h,Y}\mathbf{L}_{h}\mathbf{u}_{h,Y}, \quad \mathbf{P}_{\mathcal{N}}: \ \Xi_{h} \longrightarrow U_{h,Y},$$

where  $\mathbf{u}_{h,Y} \in U_{h,Y}$  in (5.2) is such that  $\mathbf{Tr}_{h}\mathbf{u}_{h,Y} = \xi_{h}$  and arbitrary in the rest. It is shown in [8] that the potential  $\mathbf{P}_{\mathcal{N}}\xi_{h}$  depends only on  $\xi_{h}$  and not on the choice of  $\mathbf{u}_{h,Y}$ in the formula (5.2). Therefore, we can choose  $\mathbf{u}_{h,Y} \in U_{h,Y}$ ,  $\mathbf{Tr}_{h}\mathbf{u}_{h,Y} = \xi_{h}$ , in (5.2) in such a way that it will turn into zero everywhere except in some neighborhood of the set  $\gamma$  or even simply everywhere except  $\gamma$ . Then, since the operator  $\mathbf{L}_{h}$  acts according to local formulae, the function  $\mathbf{L}_{h}\mathbf{u}_{h,Y} \in F_{h,Y}$  will, generally speaking, differ from zero also in some small neighborhood of  $\gamma$  (we mean here the "grid neighborhood," i.e., neighboring nodes). Thus, we can really consider only the functions with compact support as the right-hand sides for AP (see page 1360). Evidently, the potential  $\mathbf{P}_{\mathcal{N}}$  satisfies the boundary conditions of AP (4.3), (4.13), and it is a solution of the homogeneous equation  $\mathbf{L}_{h}\mathbf{u}_{h,Y} = \mathbf{0}$ .

Further, introduce the difference boundary projection [8]

(5.3) 
$$\mathbf{P}_{\gamma}: \Xi_h \longrightarrow \Xi_h, \quad \mathbf{P}_{\gamma} \stackrel{\text{def}}{=} \mathbf{Tr}_h \mathbf{P}_{\mathcal{N}},$$

The following proposition takes place for  $\mathbf{P}_{\gamma}$  (see [8]): the equality

(5.4) 
$$\xi_h - \mathbf{P}_{\gamma} \xi_h = \mathbf{0}$$

is valid for those and only those  $\xi_h \in \Xi_h$  which are the trace  $\xi_h = \mathbf{Tr}_h \mathbf{u}_{h,Y}$  of some solution  $\mathbf{u}_{h,Y} \in U_{h,Y}$  to the homogeneous equation  $\mathbf{L}_h \mathbf{u}_{h,Y} = \mathbf{0}$ . In the case when (5.4) holds, the equation  $\mathbf{L}_h \mathbf{u}_{h,Y} = \mathbf{0}$  has a unique solution  $\mathbf{u}_{h,Y} \in U_{h,Y}$  with trace  $\xi_h$ ,  $\mathbf{Tr}_h \mathbf{u}_{h,Y} = \xi_h$ . This solution can be computed by means of the generalized difference Green formula [8]

(5.5) 
$$\mathbf{u}_{h,Y} = \mathbf{P}_{\mathcal{N}} \xi_h$$

Thus, we have reduced the difference boundary-value problem in  $D_Y$  to equation (5.4) at the grid boundary  $\gamma$ . Now let us show how one can use (5.4) for constructing the ABCs.

Assume for simplicity that we are solving the Navier–Stokes equations in  $D_{in}$  by means of some finite-difference technique using the "O"-type grid. The curve  $\Gamma$  is the last but one closed coordinate line of this grid. Designate as  $\nu$  the set of "O"-type grid nodes belonging to  $\Gamma$ .  $\Gamma_1$  is the last closed coordinate line;  $\nu_1$  is the set of nodes belonging to  $\Gamma_1$ . Let the space stencil of the scheme used in  $D_{in}$  be not more than  $3 \times 3$ . Then, to obtain a complete system of difference equations in  $D_{in}$ , one essentially requires some additional relations between the unknowns in the nodes  $\nu$  and  $\nu_1$  since the center of the scheme stencil cannot coincide with any of the  $\nu_1$  nodes. These additional relations will be obtained from the solution of the linear problem since all the nodes  $\nu_1$  already belong to that region where the flow is governed by the linear equations. That is, we consider the parameters at  $\Gamma$  (more exactly, in the nodes  $\nu$ ) as the known values and using them as boundary data for the external linear problem in  $D_{\text{ex}}$  we find the corresponding parameters in the nodes  $\nu_1$ .

Recall that in §2 we introduced the space  $\Xi$  of continuous clear traces consisting of vector functions  $\xi$  with the components representing the values of the solution and its normal derivatives at  $\Gamma$ . Now introduce some finite-dimensional approximation to the space  $\Xi$  and designate it  $\Xi_{\omega} \ni \xi_{\omega}$ . We can approximate functions  $\xi \in \Xi$  with the help of local splines [8] depending on the values in some finite set of base points  $\omega \subset \Gamma$ , then  $\xi_{\omega}$  are the values in these points. It is also possible to use trigonometrical polynomials, then  $\xi_{\omega}$  are the vector functions the finite-dimensional approximation is implemented componentwise. The dimensionality of  $\Xi_{\omega}$  is  $8|\omega|$  where  $|\omega|$  corresponds to each component. If  $\mathbf{R} : \Xi_{\omega} \longrightarrow \Xi$  is an operator of spline or trigonometric interpolation then we will certainly require the fulfillment of the approximation property:  $\forall \epsilon > 0$  there exists a set  $\omega$  of sufficiently large dimension  $|\omega|$  such that  $\forall \xi \in \Xi \exists \xi_{\omega} \in \Xi_{\omega} : \\ \|\xi - \mathbf{R}\xi_{\omega}\|_{\Gamma} < \epsilon$  where  $\|\cdot\|_{\Gamma}$  is the norm chosen in an appropriate way (see [8], [7]).

Since  $\xi_{\omega}$  contains information both about the functions  $u, v, p, \rho$  themselves (i.e., perturbations with respect to the free stream background), and about their normal derivatives at  $\Gamma$ , one can easily compute  $\xi_h$  using the first two terms of the Taylor formula (recall that  $\xi_h$  is the 4-component vector function containing only  $u, v, p, \rho$  themselves and the points  $\gamma$  where  $\xi_h$  is defined are located near  $\Gamma$ ). We introduce the special operator  $\pi : \Xi_{\omega} \longrightarrow \Xi_h$  to designate the procedure of boundary data continuation from the boundary to the domain.

Let us note here that data continuation from the boundary to the domain (i.e., from  $\omega$  to  $\gamma$ ) is actually one of the main elements of the DPM [8] when applied to the solution of boundary-value problems. The procedure of data continuation is always based on the Taylor formula and therefore involves boundary values of the solution and its normal derivatives. We now make the following remark.

*Remark.* The construction of a clear trace  $\Xi$  involving unknown functions and their normal derivatives at  $\Gamma$  which is used in this paper (see §2) is not a unique possibility. It is shown in [8] that for any specific problem one can define  $\Xi$  and potential in different ways. Of course, constructions of clear traces and potentials should be correlated to each other, they then possess the main property; namely, boundary equation with projection is equivalent to the original equation on the domain.

For our current purposes the construction of  $\Xi$  used here (§2) is presumably the most convenient. Indeed, our final goal is simply to complete the system of difference equations in  $D_{\rm in}$ , i.e., to relate the values of the solution at inner ( $\Gamma$ ) and outermost ( $\Gamma_1$ ) coordinate lines. To do this, we have to represent the solution outside  $\Gamma$  (i.e., outside  $D_{\rm in}$ ) in the form of a generalized potential depending on  $(u, v, p, \rho)|_{\Gamma}$  and then to use it for determining  $(u, v, p, \rho)|_{\Gamma_1}$ . Therefore, we actually need "a resolved form of the projection," i.e., an operator expressing normal derivatives in terms of functions (see below). Using this operator and the Taylor formula we will be able to continue any specific data from  $\Gamma$  to  $\gamma$ , and then from  $\gamma$  to  $\Gamma_1$  using the potential itself as well as some interpolation procedure (see below).

We also note that many authors develop and use the so-called Poincaré–Steklov operators, mainly for domain decomposition techniques (see, e.g., [17] and references there). These operators are similar in structure to our "resolved projections" although their means of derivation are actually less general.

As was previously mentioned, we consider all the parameters  $(u, v, p, \rho)^T|_{\nu} \equiv \mathbf{u}_{\nu} \in U_{\nu}$  in the  $\nu$  nodes as known values. Now let us construct a special interpolation

operator  $\mathbf{R}_{\nu} : U_{\nu} \longrightarrow \Xi_{\omega}$  (depending on the structure of  $\Xi_{\omega}$  space; convergence is evidently required in doing so).  $\mathbf{R}_{\nu} \mathbf{u}_{\nu}$  yields those components of  $\xi_{\omega}$  which contain the values of functions themselves but not the derivatives (i.e., a half of  $\xi_{\omega}$  components). Namely, if  $\xi_{\omega} = (\xi_{\omega}^{(1)}, \xi_{\omega}^{(2)})^T$  where the vector  $\xi_{\omega}^{(1)}$  contains the functions  $(u, v, p, \rho)$ and  $\xi_{\omega}^{(2)}$  contains their normal derivatives at  $\Gamma$  then  $\xi_{\omega}^{(1)} = \mathbf{R}_{\nu}\mathbf{u}_{\nu}$ . Further, by applying the operator  $\pi$ :  $\xi_h = \pi \xi_{\omega} \stackrel{\text{def}}{=} \pi^{(1)}\xi_{\omega}^{(1)} + \pi^{(2)}\xi_{\omega}^{(2)} = \pi^{(1)}\mathbf{R}_{\nu}\mathbf{u}_{\nu} + \pi^{(2)}\xi_{\omega}^{(2)}$  and substituting this expression into (5.4), we get

(5.6) 
$$\mathbf{Q}^{(1)}\mathbf{u}_{\nu} + \mathbf{Q}^{(2)}\xi_{\omega}^{(2)} = 0,$$

where  $\mathbf{Q}_{\gamma} \stackrel{\text{def}}{=} \mathbf{I} - \mathbf{P}_{\gamma}$ , **I** is an identity operator,  $\mathbf{Q}^{(1)} = \mathbf{Q}_{\gamma} \pi^{(1)} \mathbf{R}_{\nu}$ ,  $\mathbf{Q}^{(2)} = \mathbf{Q}_{\gamma} \pi^{(2)}$ . Equation (5.6) with respect to  $\xi_{\omega}^{(2)}$  is, generally speaking, overdetermined and has no solution. We define its generalized solution in the sense of the least squares method, introducing some Euclidean norm  $\|\cdot\|_{\gamma}$  in  $\Xi_h$  for this purpose. Namely, let

$$\|\xi_h\|_{\gamma}^2 = \sum_{l=1}^4 \|\xi_h^l\|_{\gamma}^2,$$

where  $\xi_h^l$  is the *l*th component of the vector function  $\xi_h$ . We define the norm  $\|\xi_h^l\|_{\gamma}$  as follows:

$$\left\| \xi_{h}^{l} \right\|_{\gamma}^{2} = \frac{1}{|\gamma|} \sum_{j \in \gamma} \left[ \left| \xi_{h_{j}}^{l} \right|^{2} + \sum_{\substack{j' \in St_{\gamma_{j}}^{5} \cap \gamma \\ j' \neq j}} \left| \frac{\xi_{h_{j}}^{l} - \xi_{h_{j'}}^{l}}{\varrho_{j,j'}} \right|^{2} \right].$$

where j is a subscript enumerating nodes from  $\gamma$ ,  $|\gamma|$  is the total number of nodes in  $\gamma$ ,  $St_{\gamma_j}^5$  is a five-node stencil "right cross" with its center at  $\gamma_j$ , and  $\varrho_{j,j'}$  is the distance between  $\gamma_j$  and  $\gamma_{j'}$ .

Introduce the scalar product in  $\Xi_h$  putting for  $a, b \in \Xi_h$ ,

$$(a, b) = \sum_{l=1}^{4} \sum_{j \in \gamma} a_j^l b_j^l.$$

In doing so, one can specify a symmetric linear operator  $\mathbf{A}_{\gamma}$ :  $\Xi_h \longrightarrow \Xi_h$  such that the norm introduced above would be written in the form

$$\left\| \xi_h \right\|_{\gamma}^2 = \left( \mathbf{A}_{\gamma} \xi_h, \xi_h 
ight).$$

Define the generalized solution to (5.6) as the solution to the variational problem

(5.7) 
$$\left\| \mathbf{Q}^{(1)} \mathbf{u}_{\nu} + \mathbf{Q}^{(2)} \boldsymbol{\xi}_{\omega}^{(2)} \right\|_{\gamma}^{2} \longrightarrow \min$$

The necessary conditions of minimum for (5.7) (the Lagrange–Euler equations) are provided by the following linear system:

(5.8) 
$$\mathbf{Q}^{(2)^T} \mathbf{A}_{\gamma} \mathbf{Q}^{(2)} \boldsymbol{\xi}_{\omega}^{(2)} = -\mathbf{Q}^{(2)^T} \mathbf{A}_{\gamma} \mathbf{Q}^{(1)} \mathbf{u}_{\nu}$$

to be solved with respect to  $\xi_{\omega}^{(2)}$ , i.e., actually with respect to the normal derivatives of solution **u** at  $\Gamma$ .

One can find the solution to (5.8) by means of a direct technique (see below):

(5.9) 
$$\xi_{\omega}^{(2)} = \mathbf{K}_{\omega}^{-1} \mathbf{K}_{\nu} \mathbf{u}_{\nu}$$

$$\mathbf{K}_{\omega} \stackrel{\text{def}}{=} \mathbf{Q}^{(2)^{T}} \mathbf{A}_{\gamma} \mathbf{Q}^{(2)}, \quad \mathbf{K}_{\nu} \stackrel{\text{def}}{=} - \mathbf{Q}^{(2)^{T}} \mathbf{A}_{\gamma} \mathbf{Q}^{(1)},$$

and then obtain

(5.10) 
$$\xi_h = \pi^{(1)} \mathbf{R}_\nu \mathbf{u}_\nu + \pi^{(2)} \mathbf{K}_\omega^{-1} \mathbf{K}_\nu \mathbf{u}_\nu.$$

Note that the operator  $\mathbf{K}_{\omega}^{-1}\mathbf{K}_{\nu}$  from (5.9) is actually "the resolved form of projection" we were discussing above. We use quotation "." here since  $\mathbf{K}_{\omega}^{-1}\mathbf{K}_{\nu}$  itself is no longer a projection. It relates the values of functions and normal derivatives at  $\Gamma$  and therefore allows us to obtain  $\xi_h$  on the basis of  $\mathbf{u}_{\nu}$  (formula (5.10)).

We then have to find the solution in  $\nu_1$ -nodes. To do this we will use formula (5.5) with slight modifications. Namely, we will find the values  $\mathbf{u}_{\nu_1}$  with the help of interpolation from the grid  $\mathcal{N}$ . Designate as  $\kappa$  that subset of  $\mathcal{N}$  nodes where it is necessary to know the solution  $\mathbf{u}_{h,Y} = \mathbf{P}_{\mathcal{N}}\xi_h$  in order to implement an interpolation procedure of sufficiently high order, e.g., interpolation by quadratic polynomials. Evidently,  $\kappa$  is the grid set located near  $\Gamma_1$ . Write instead of (5.5),

$$\mathbf{u}_{\nu_1} = \mathbf{P}_{\nu_1} \xi_h,$$

where  $\mathbf{P}_{\nu_1} \stackrel{\text{def}}{=} \mathbf{R}_{\nu_1 \kappa} \mathbf{P}_{\mathcal{N}}, \mathbf{R}_{\nu_1 \kappa}$  is an interpolation operator from  $\kappa$  to  $\nu_1$ . Substituting (5.10) into (5.11) we get

(5.12) 
$$\mathbf{u}_{\nu_1} = \mathbf{P}_{\nu_1} \left( \pi^{(1)} \mathbf{R}_{\nu} + \pi^{(2)} \mathbf{K}_{\omega}^{-1} \mathbf{K}_{\nu} \right) \mathbf{u}_{\nu} = \mathbf{T} \mathbf{u}_{\nu}.$$

The relation (5.12) yields the connection between  $\mathbf{u}_{\nu}$  and  $\mathbf{u}_{\nu_1}$  required for completion of the system of difference equations in  $D_{in}$ . Let us now point out the following important circumstance. All the operators considered in this section, in particular  $\mathbf{A}_{\gamma}, \mathbf{Q}_{\gamma}, \mathbf{K}_{\omega}, \mathbf{K}_{\nu}, \mathbf{R}_{\nu}, \pi, \mathbf{R}_{\nu_{1}\kappa}, \mathbf{P}_{\nu_{1}}, \mathbf{T}, \text{ act in finite-dimensional spaces and conse$ quently can be computed in the form of matrices with respect to corresponding bases. (Then, in particular, it is possible to solve (5.8) by means of a direct method.) The computation of these matrices requires, generally speaking, the repeated  $(8|\omega|)$  solution of difference AP. However, this procedure (solution of AP in order to compute  $\mathbf{Q}_{\gamma}$  and  $\mathbf{P}_{\nu_1}$ ) does not demand a lot of computer time (even without fast Fourier transform usage) since the right-hand sides are concentrated near  $\gamma$  (see page 1377) and the solution is to be known only in the nodes  $\gamma \cup \kappa$ . These subsets of the grid  $\mathcal{N}^0$ are located not far from the artificial boundary and the number of nodes in  $\gamma$  and  $\kappa$ does not depend on Y while  $h_x$ ,  $h_y$ , and X are constant. Therefore, the number of operations required here for the implementation of both direct and inverse discrete Fourier transforms is  $O(M \cdot J)$ , and no longer  $O(M \cdot J^2)$  as in the case of solving the problem on the whole grid  $\mathcal{N}^0$ . On the other hand, the computation of matrix  $\mathbf{T}$  provides significant advantages from the viewpoint of the numerical realization of these ABCs. For example, if one uses some explicit pseudo-time integration technique for the solution of the Navier–Stokes equations inside  $D_{in}$  then the matrix relation (5.12) is simply applied at each iteration to complement the values  $\mathbf{u}_{\nu_1}$ . If the stencil

of scheme in  $D_{in}$  is more than  $3 \times 3$  then one has to compute **u** with the help of a potential not only on  $\nu_1$  but also on the extra one or few coordinate lines external to  $\Gamma$ . The computation technique itself is not changed in such a case.

Note, in addition, that since we use the difference potential (5.2) (see also (5.5)) for solving the linear problem then the questions of approximation and convergence are of great importance. We mean here an approximation to a continuous potential by the difference potential and convergence of the difference periodic solution to the continuous periodic solution (it is "a half" of Definition 3.2) while the grid size hvanishes and  $|\omega|$  consistently grows (see [8]). These questions are studied in detail in [8]. In particular, it is shown there that the type of norm providing convergence and the convergence rate depend on the order of approximation to the differential operator  $\mathbf{L}$  by the difference operator  $\mathbf{L}_h$  and also on the order of the Taylor formula in the operator  $\pi$ . In our case, one can expect convergence of the difference solution together with the first difference derivatives [8].

6. Computational results. In this section, we present the results of our numerical experiments. We begin with some model computations related to the estimates from §3 as well as to the remark on page 1373. Then we describe some results of computation of the viscous flow past an airfoil.

**6.1. Eigenvectors of Q**<sub>0</sub>. For the various flow regimes considered (differing by the values of Ma and Re) and for various grids, the matrix  $\mathbf{Q}_0$  (see (4.12)) has a quintuple eigenvalue with the unit module:  $|\mu_s(0)| = 1$ ,  $s = 3, \ldots, 7$ ,  $(|\mu_1(0)| < 1, |\mu_2(0)| < 1, |\mu_8(0)| > 1$ ). As was previously mentioned, for all the remaining  $k \neq 0$  the absolute values of all the eigenvalues differ from 1. Special computations show that there exist five linearly independent eigenvectors corresponding to  $|\mu_s(0)| = 1$ ,  $s = 3, \ldots, 7$ ; i.e., the remark on page 1373 is valid. This verification was carried out as follows. The whole space  $C^8$  of the solutions to the homogeneous system (4.8) (for k = 0) can be represented as a direct sum  $C^8 = C_1 \oplus C_1$  where

(6.1) 
$$\forall \mathbf{v} \in C_1 : \prod_{|\mu_s(0)|=1} (\mathbf{Q}_0 - \mu_s(0)\mathbf{I}) \mathbf{v} = \mathbf{0},$$
$$\forall \mathbf{v} \in C_{\overline{1}} : \prod_{|\mu_s(0)|\neq 1} (\mathbf{Q}_0 - \mu_s(0)\mathbf{I}) \mathbf{v} = \mathbf{0}.$$

The proof of this statement is the same as that of the formulae (4.13), (4.14) (see Appendix (§7)). Then, an arbitrary element  $\mathbf{v}_0 \in C^8$  is chosen and the powers of the  $\mathbf{Q}_0$  matrix are applied to this element, and the projection onto the subspace  $C_1$  is implemented at each step

(6.2) 
$$\mathbf{v}'_m = \mathbf{Q}_0 \mathbf{v}_{m-1}, \ m = 1, 2, \dots,$$

(6.3) 
$$\mathbf{v}_m = \mathbf{v}'_m - \mathbf{v}''_m \text{ where } \mathbf{v}_m \in C_1, \ \mathbf{v}''_m \in C_{\bar{1}}.$$

The representation (6.3) is evidently unique. The computations show that the estimate  $\|\mathbf{v}_m\| \leq \text{const}$  holds for sufficiently large  $m \sim 10^3 \div 10^4$ . It confirms that there is now polynomial growth and consequently that there are no Jordan blocks of order more than 1 corresponding to  $|\mu_s(0)| = 1, s = 3, \ldots, 7$  in the canonical form of the matrix  $\mathbf{Q}_0$ .

6.2. Computation of the operator **T**. Now let us consider some results of calculating the operators **T** from (5.12). We carried out these computations for Ma = 0.5, Re = 500, Pr = 0.72;  $\Gamma$  and  $\Gamma_1$  are concentric circles with radii r = 1.0 and  $r_1 = 2.4$ , respectively, with center  $(0, \pi)$ ,  $X = 2\pi$ ; the grid of AP consists of square cells with sizes  $h_x = h_y = 2\pi/M$ , M is an integer number; sets  $\nu$  and  $\nu_1$  consist of  $|\nu| = |\nu_1| = 25$  nodes. We recall here that the operator **T** should connect the values of the solution at the inner and outermost coordinate lines of the grid used inside  $D_{in}$ . The particular choice of  $\Gamma$  and  $\Gamma_1$  here models these two coordinate lines, respectively; see §5 for details. The parameters M, Y, as well as  $|\omega|$  (the latter is the dimension of finitely dimensioned space  $\Xi_{\omega}$  approximating  $\Xi$ ; see page 1377), were varied for different computation variants.

The series of computations were performed for M = 17 and M = 33 in order to clarify the character of the operator **T** (see (5.12)) dependence on the value of period Y. Namely, the sequence of problems for  $Y_{s+1} = \beta Y_s$ ,  $s = 1, 2, \ldots$ ,  $Y_1 = X$ ,  $\beta = 2$ , was solved and the behavior of the value

(6.4) 
$$\sigma_s = \frac{\|\mathbf{T}_{s+1} - \mathbf{T}_s\|}{0.5 \left(\|\mathbf{T}_{s+1}\| + \|\mathbf{T}_s\|\right)}$$

as depends on s was studied.  $\sigma_s$  from (6.4) evidently characterizes the variation of **T** caused by the change of period. Grid sizes in this case remain constant. The operator norm in (6.4) is induced by the norms of functions  $\mathbf{u}_{\nu}$  and  $\mathbf{u}_{\nu_1}$ ; the Hilbert norm  $l_2$  was used while computing. Since we consider sufficiently smooth solutions it is possible to assume that the dependence of  $\sigma_s$  on s has the same character as the dependence of the value  $\|\mathbf{u}_{h,Y_{s+1}}^0 - \mathbf{u}_{h,Y_s}^0\|_{\check{y}}$  on s (see designations of §3). By virtue of the evident inequality

$$\left\|\mathbf{u}_{h,Y_{s+1}}^{0}-\mathbf{u}_{h,Y_{s}}^{0}\right\|_{\breve{y}} \leq \left\|\mathbf{u}_{h,Y_{s+1}}^{0}-\left[\mathbf{u}^{0}\right]_{h}\right\|_{\breve{y}}+\left\|\mathbf{u}_{h,Y_{s}}^{0}-\left[\mathbf{u}^{0}\right]_{h}\right\|_{\breve{y}}$$

and the same arguments used to derive (3.16), the following estimate takes place:

(6.5) 
$$\left\| \mathbf{u}_{h,Y_{s+1}}^{0} - \mathbf{u}_{h,Y_{s}}^{0} \right\|_{\breve{y}} \leq \frac{\beta+1}{\beta} \epsilon_{\mathrm{opt}}(Y_{s}) + \left( c_{Y_{s}} + c_{Y_{s+1}} \right) |h|^{q}$$

The right-hand side of (6.5) depends on Y and on h in the same way as  $\epsilon_{opt}(h, Y)$  does; see formulae (3.16) and (3.17). Moreover, if  $c_Y = c_5 = \text{const}$  (see page 1368) then the first term on the right-hand side of (6.5) characterizes the influence of period and the second one is responsible for the influence of grid size. The values of  $\sigma_s$  for the different parameters used are presented in Table 6.1 (recall that  $h_x$  and  $h_y$  do not depend on s).

A monotonic decrease of  $\sigma_s$  in all the computations justifies that the assumption  $c_Y = c_5 = \text{const}$  is true. In the opposite case when  $c_Y$  increases with an increase of Y one has to first observe a decrease and then a growth of  $\sigma_s$ .

The values of  $\sigma_s$  are sufficiently small for large Y which is presumably caused by the phenomenon of convergence (also on the finite set  $(0, X) \times (-\check{y}, \check{y})$ ) of the difference periodic solution to the difference nonperiodic one, i.e., to the solution formally determined in  $D^0$  on the infinite grid with constant size. As for the difference between each difference solution and the corresponding continuous one, it remains the same and is determined by the grid size h. Therefore it makes no sense to choose too large a Y for the fixed h. As regards size h itself of the grid  $\mathcal{N}^0$ , its value is to be connected with the size of the grid used inside  $D_{\rm in}$  for the integration of the Navier– Stokes equations. Namely, the accuracy of the solution to the linear problem is to

s	M = 17	M = 33		
	$ \omega  = 19$	$ \omega  = 19$	$ \omega  = 25$	
1	0.345	0.361	0.290	
2	0.212	0.267	0.206	
3	$7.556 \cdot 10^{-2}$	0.158	0.125	
4	$4.794 \cdot 10^{-2}$	$9.191 \cdot 10^{-2}$	$7.296 \cdot 10^{-2}$	
5	$3.893 \cdot 10^{-2}$	$4.552 \cdot 10^{-2}$	$3.558\cdot10^{-2}$	
6	$2.936\cdot10^{-2}$			

ΤA	BLE	6.1	

be at least not worse than the accuracy of the solution to the nonlinear one. An application experience of the analogous ABCs for the Euler equations [3], [11] shows that it is always sufficient to choose h in such a way that the average distance between  $\Gamma$  and  $\Gamma_1$  would be about several cells of the grid  $\mathcal{N}^0$ . Concerning the value  $|\omega|$ , it is shown in [8] that once the quadratic interpolation is used (operators  $\pi$  and  $\mathbf{R}_{\nu}$ ; see pages 1378, 1378) then the relation  $h \sim |\omega|^{-2}$  has to hold; i.e., one ought to increase  $|\omega|$  consistently with the grid  $\mathcal{N}^0$  refinement. However, it should be noted here that if the grid in  $D_{\rm in}$  is prescribed then it is not worth choosing  $|\omega| > |\nu|$ .

6.3. Viscous flow past an airfoil. Here we are going to present and discuss some results of our ABCs' implementation to computations of real viscous flows. Namely, we study a gas flow past the NACA0012 airfoil for Mach number at infinity Ma = 0.63 and attack angle  $\alpha = 2^{\circ}$ . This regime is well known while being inviscid. It is studied in numerous papers including [11]; see also the bibliography in [3] as well as the reviews [1], [2]. However, here we compute this flow for the laminar viscous regime Re = 4000. From the pure gasdynamic viewpoint it is a subcritical, i.e., fully subsonic flow, and once the viscosity is introduced the flow also appears to be separated (a separation zone is located near the trailing edge). To integrate the Navier-Stokes equations inside  $D_{in}$  we use a pseudo-time multigrid iteration procedure realized in the finite-volume code [18]-[20]. The computations are implemented on the C-type curvilinear boundary-fitted grid of  $256 \times 64$  nodes generated around the airfoil. An "average radius" of computational domain for this specific case is about 5.5 chords of the airfoil. In doing so, both  $\Gamma$  and  $\Gamma_1$  are nonsmooth (each has two "corner points"). However, experience at solving the boundary-value problems by means of the DPM in the domains with nonsmooth boundaries [8] provides us with reasons to calculate  $\mathbf{T}$  for this case exactly as was described above. The only slight difference is in the construction of the operator  $\pi$  (page 1378) of boundary data continuation near these "break points," but it is not essential for current consideration.

While integrating the Navier–Stokes equations we use four levels of multigrid with W-cycles and implement the ABCs (5.12) at each iteration only on the finest level; for the coarser levels we retain the boundary values provided by the finest one. We compare our results with those obtained while implementing standard external boundary conditions included in the code [18]–[20]. The latter are based on the analysis of characteristic variables at inflow and extrapolation at outflow. An advantage of these conditions is their algorithmic simplicity as well as very low computational expenditure.

In Fig. 6.1 we present the convergence dynamics (dependence of  $\rho$ -residual in the *C*-norm on the number of iterations) for this computation. We use three different op-



F1G. 6.1. Convergence dynamics for NACA0012:  $\alpha = 2^{\circ}$ , Ma = 0.63 Re = 4000. Logarithm of  $\rho$ -residual in C-norm versus number of iterations.

erators **T** which correspond to three curves marked nonlocal 2, nonlocal 3, nonlocal 4. The number (2,3,4) here represents the value of period Y in units of computational domain diameter. It turns out that it is more convenient to measure Y in these units and not in airfoil chords.

One can see that for all the cases where nonlocal ABCs are implemented the theoretical convergence rate is more than three times faster than for extrapolation. (The theoretical convergence rate is just a number of iterations required to reduce the initial residual by a prescribed factor.) Of course, we also have to take into account the additional computational expenditure caused by the nonlocal nature of these ABCs. This additional expenditure consists of two parts. The first one is the CPU time required for matrix-vector multiplication (see (5.12)) at each iteration, with each iteration becoming about 10% more expensive. The second is the CPU time required for the computation of  $\mathbf{T}$  itself. This part, of course, depends on Y. It turns out that for the specific case under investigation the computation of operator  $\mathbf{T}_2$  requires about 52' of CPU time IBM RISC 6000/540, the operator  $\mathbf{T}_3$  requires about 80', and  $T_4$  requires about 120'. Now compare these figures with the CPU time required for the integration of the Navier–Stokes equations inside  $D_{in}$ . One iteration "costs" about 14.9'' for the simplest extrapolation conditions (therefore, about 16.4''for the nonlocal ABCs). If we assume that the accuracy  $10^{-8}$  is satisfactory (which is natural) then we need 4600 usual iterations, which implies about 19 hours of CPU time and only 1500 iterations with nonlocal ABCs, which means 6 hours 49' and an additional 52' for the  $T_2$  computation. The integral gain in convergence acceleration still remains slightly less than three times, which is most essential.

We also have to analyze accuracy, namely, how the type of ABCs influences the solution inside  $D_{\rm in}$ . Table 6.2 contains the values of dynamic force coefficients ( $C_l$  — lift and  $C_d$  — drag) for the computations described above.

	Extrapolation	$\mathbf{T}_2$	$\mathbf{T}_3$	$\mathbf{T}_4$
$\overline{C}_l$	0.02509	0.02455	0.02470	0.02470
$C_d$	0.03129	0.03147	0.03144	0.03139

TABLE 6.2

One can easily see that the difference between the corresponding coefficients is very slight from which we conclude the following. First, it is quite sufficient to use only "cheap" operators  $\mathbf{T}$  being computed for small values of Y. Second, all the solutions obtained in the case of nonlocal ABCs' implementation are reasonable which in particular justifies the possibility of linearization (at least for this specific case).

We do not present here the results of other viscous flow computations. We have been studying various flow regimes including turbulent and transonic ones. In addition to drastic convergence acceleration we have found that while using nonlocal ABCs it is possible to essentially shrink the computational domain preserving the accuracy of computations.

We discuss the computational results in detail as well as some related topics and generalizations in a new paper [21].

7. Appendix. Consider a linear space  $C^n$  of *n*-dimensional vectors with complex components and some linear operator  $\mathbf{Q} : C^n \longrightarrow C^n$  acting in this space. Let  $C \ni \mu_s$ ,  $s = 1, \ldots, \bar{s}$  be all the different eigenvalues of operator  $\mathbf{Q}$ ,  $\bar{s} \le n$ ,  $n_s$  are the multiplicities of these eigenvalues,  $\sum_{s=1}^{\bar{s}} n_s = n$ . Moreover, let  $\mathbf{e}_s^r \in C^n$ ,  $s = 1, \ldots, \bar{s}$ ,  $r = 1, \ldots, r_s, 1 \le r_s \le n_s$  be all the linearly independent eigenvectors of the mapping  $\mathbf{Q}$ :

(7.1) 
$$\mathbf{Q}\mathbf{e}_{s}^{r}=\mu_{s}\mathbf{e}_{s}^{r},$$

 $\bar{s} \leq \sum_{s=1}^{\bar{s}} r_s \leq n$ . For the case  $r_s < n_s$  the adjoint vectors  $\mathbf{e}_s^{r,p}$ ,  $2 \leq p \leq p_s^r$ , of the mapping  $\mathbf{Q}$  exist for some  $\mathbf{e}_s^r$ ,  $1 \leq r \leq r_s$ :

(7.2) 
$$\mathbf{Q}\mathbf{e}_s^{r,p} = \mu_s \mathbf{e}_s^{r,p} + \mathbf{e}_s^{r,p-1}, \ 2 \le p \le p_s^r, \ \mathbf{e}_s^{r,1} \stackrel{\text{def}}{=} \mathbf{e}_s^r.$$

Here  $p_s^r$  is the order of the Jordan block corresponding to the eigenvector  $\mathbf{e}_s^r \equiv \mathbf{e}_s^{r,1}$  in the canonical form of the matrix  $\mathbf{Q}$ . The following relation holds:  $\sum_{r=1}^{r_s} p_s^r = n_s \Rightarrow \sum_{s=1}^{\bar{s}} \sum_{r=1}^{r_s} p_s^r = n$ . The system of vectors  $\{\mathbf{e}_s^{r,p} | 1 \leq s \leq \bar{s}, 1 \leq r \leq r_s, 1 \leq p \leq p_s^r\}$  is the basis in  $C^n$ ; the matrix  $\mathbf{Q}$  has a canonical Jordan form just in this basis.

Consider the linear span of all those elements of the basis  $\{\mathbf{e}_s^{r,p}\}$  corresponding to certain s; namely,  $C^{n_s} = \ln \{\mathbf{e}_s^{r,p} | 1 \leq r \leq r_s, 1 \leq p \leq p_s^r\}$ .  $C^{n_s}$  is the subspace of dimension  $n_s$  in  $C^n$ . Moreover, by virtue of (7.1) and (7.2),  $C^{n_s}$  is an eigensubspace of the operator  $\mathbf{Q}$ ; i.e.,  $\forall \mathbf{v} \in C^{n_s}$ :  $\mathbf{Q} \mathbf{v} \in C^{n_s}$ . Note that the constructions of  $C^{n_s}$  guarantee that  $C^{n_{s_1}} \cap C^{n_{s_2}} = 0$  if  $s_1 \neq s_2$ . Evidently, one can represent the whole subspace  $C^n$  as the direct sum of the subspaces  $C^{n_s}$  corresponding to different eigenvalues of the operator  $\mathbf{Q}$ :

$$C^n = \bigoplus_{s=1}^{\bar{s}} C^{n_s}$$

Define the operator  $\mathbf{S}_s \stackrel{\text{def}}{=} (\mathbf{Q} - \mu_s \mathbf{I})^{n_s}$  where  $\mathbf{I}$  is an identity operator. Let  $\mathbf{v} \in C^{n_s}$ ; i.e.,  $\mathbf{v} = \sum_{r=1}^{r_s} \sum_{p=1}^{p_s^r} \beta_s^{r,p} \mathbf{e}_s^{r,p}$ ,  $\beta_s^{r,p} \in C$ . Since  $p_s^r \leq n_s - (r_s - 1)$  always, then by

virtue of (7.1) and (7.2),  $\mathbf{S}_s \mathbf{e}_{s'}^{r,p} = \mathbf{0}$ ,  $r = 1, \ldots, r_s$ ,  $p = 1, \ldots, p_s^r$ , and, consequently,  $\mathbf{S}_s \mathbf{v} = \mathbf{0}$ . Conversely, now let  $\mathbf{v} \in C^n$  be some vector satisfying the condition  $\mathbf{S}_s \mathbf{v} = \mathbf{0}$ . Expanding  $\mathbf{v}$  in terms of the basis  $\{\mathbf{e}_s^{r,p}\}$ :  $\mathbf{v} = \sum_{s'=1}^{\bar{s}} \sum_{r=1}^{r_{s'}} \sum_{p=1}^{p_{s'}} \beta_{s'}^{r,p} \mathbf{e}_{s'}^{r,p}$  and applying the operator  $\mathbf{S}_s$  to this expansion we get

(7.3) 
$$\mathbf{v} = \sum_{\substack{s'=1\\s'\neq s}}^{\bar{s}} \sum_{r=1}^{r_{s'}} \sum_{p=1}^{p_{s'}^r} \beta_{s'}^{r,p} \left(\mathbf{Q} - \mu_s \mathbf{I}\right)^{n_s} \mathbf{e}_{s'}^{r,p} = \mathbf{0}.$$

Taking into account that  $\mu_{s'} \neq \mu_s$  for  $s' \neq s$  we find from (7.1) and (7.2) the following: (i)  $(\mathbf{Q} - \mu_s \mathbf{I})^{n_s} \mathbf{e}_{s'}^{r,1} = (\mu_{s'} - \mu_s)^{n_s} \mathbf{e}_{s'}^{r,1} \neq \mathbf{0};$ 

(ii) if  $n_s < p$  then  $(\mathbf{Q} - \mu_s \mathbf{I})^{n_s} \mathbf{e}_{s'}^{r,p} = \sum_{l=0}^{n_s} {n_s \choose l} (\mu_{s'} - \mu_s)^{n_s - l} \mathbf{e}_{s'}^{r,p - l} \neq \mathbf{0}$  where  ${n_s \choose l} = (n_s!)/(l! (n_s - l)!)$  are binomial coefficients;

(iii) if  $n_s \ge p$  then  $(\mathbf{Q} - \mu_s \mathbf{I})^{n_s} \mathbf{e}_{s'}^{r,p} = \sum_{l=0}^{p-1} {n_s \choose l} (\mu_{s'} - \mu_s)^{n_s - l} \mathbf{e}_{s'}^{r,p - l} \ne \mathbf{0}$ . Therefore all the coefficients  $\beta_{s'}^{r,p}$  in (7.3) are equal to zero:  $\beta_{s'}^{r,p} = 0$ ,  $s' \ne s$ ,  $r = 1, \ldots, r_{s'}, p = 1, \ldots, p_{s'}^{r,i}$ ; i.e.,  $\mathbf{v} \in C^{n_s}$ .

Thus we have proved the following lemma.

LEMMA 7.1. The condition  $\mathbf{S}_s \mathbf{v} = \mathbf{0}$  is necessary and sufficient for the inclusion  $\mathbf{v} \in C^{n_s}$ .

COROLLARY 7.2. Let  $\mathbf{v} \in C^{n_{s_1}}$ ,  $1 \leq s_1 \leq \bar{s}$ . Then the inclusion  $\mathbf{S}_{s_2}\mathbf{v} \in C^{n_{s_1}}$ takes place for an arbitrary  $s_2 \in \{1, \ldots, \bar{s}\}$ . Moreover, if  $\mathbf{v} \neq \mathbf{0}$  then  $\mathbf{S}_{s_2}\mathbf{v} \neq \mathbf{0}$  for  $s_2 \neq s_1$ .

*Proof.*  $C^{n_{s_1}}$  is the eigensubspace of the operator  $\mathbf{Q}$ ; therefore,  $C^{n_{s_1}}$  is an eigensubspace for any operator  $(\mathbf{Q} - \mu \mathbf{I})^t \quad \forall \mu \in C, \forall t \in N$ . Consequently,  $\mathbf{S}_{s_2} \mathbf{v} \in C^{n_{s_1}}$ . If  $\mathbf{v} \neq \mathbf{0}$  and  $\mathbf{S}_{s_2} \mathbf{v} = \mathbf{0}$ , then, because of Lemma 7.1,  $\mathbf{v} \in C_{n_{s_2}}$ . According to the condition of Corollary 7.2,  $\mathbf{v} \in C^{n_{s_1}}$  where  $s_2 \neq s_1$ , which means  $\mathbf{v} \in C^{n_{s_1}} \cap C^{n_{s_2}} = \mathbf{0}$ . The contradiction obtained proves the corollary:  $\mathbf{S}_{s_2} \mathbf{v} \neq \mathbf{0}$ .

Now choose an arbitrary subset:  $\{s_q\}_{q=1}^{\bar{q}} \subset \{1, \ldots, \bar{s}\}$  and consider the subspace  $C^n$ :

(7.4) 
$$C' = \ln \left\{ \mathbf{e}_{s_q}^{r,p} \middle| q = 1, \dots, \bar{q}, \ 1 \le r \le r_{s_q}, \ 1 \le p \le p_{s_q}^r \right\}$$
$$= C^{n_{s_1}} \oplus C^{n_{s_2}} \oplus \dots \oplus C^{n_{s_q}}.$$

Evidently, C' is the eigensubspace of the operator **Q**. Construct the operator

(7.5) 
$$\mathbf{S}' = \prod_{q=1}^{\bar{q}} \mathbf{S}_{s_q} = \prod_{q=1}^{\bar{q}} \left( \mathbf{Q} - \mu_{s_q} \mathbf{I} \right)^{n_{s_q}}$$

Let  $\mathbf{v} \in C'$ ; this means that some unique set of vectors  $\{\mathbf{v}_q \in C^{n_{s_q}} | q = 1, \ldots, \bar{q}\}$  exists such that  $\mathbf{v} = \sum_{q=1}^{\bar{q}} \mathbf{v}_q$ . Then, taking into account Corollary 7.2 we obtain  $\mathbf{S'v} = \mathbf{0}$ .

Conversely, now let  $\mathbf{v} \in C^n$  be an arbitrary vector satisfying the condition  $\mathbf{S'v} = \mathbf{0}$ . Obviously, the unique expansion

(7.6) 
$$\mathbf{v} = \sum_{s=1}^{\bar{s}} \mathbf{v}_s, \quad \mathbf{v}_s \in C^{n_s}$$

exists. Applying the operator  $\mathbf{S}'$  to this expansion we get

(7.7) 
$$\sum_{\substack{s=1\\s\notin\{s_q\}_{q=1}^{\tilde{q}}}}^{\tilde{s}} \mathbf{S}' \mathbf{v}_s = \mathbf{0}.$$

By virtue of the construction of operator  $\mathbf{S}'$  (see (7.5)) and because of Corollary 7.2 each term in (7.7) belongs to a corresponding subspace  $C^{n_s}$ . Therefore, the equality (7.7) yields  $\mathbf{v}_s = \mathbf{0}$ ,  $s \notin \{s_q\}_{q=1}^{\bar{q}}$  or, in other words,  $\mathbf{v} \in C'$  (see (7.6)). Thus we come to the following lemma.

LEMMA 7.3. For  $\mathbf{v} \in C'$  (see (7.4)) it is necessary and sufficient that  $\mathbf{S'v} = \mathbf{0}$  (where the operator  $\mathbf{S'}$  is defined by the formula (7.5)).

Now let us consider the following homogeneous system (here we assume that  $det \mathbf{Q} \neq 0$ ):

$$\mathbf{v}_m + \mathbf{Q} \, \mathbf{v}_{m-1} = \mathbf{0}$$

with respect to the unknowns  $\mathbf{v}_m$ ,  $m = 0, 1, 2, \dots$  (7.8) is an analogue of the homogeneous variant of (4.8). Using the basis  $\{\mathbf{e}_s^{r,p}\}$  we can write the general solution of (7.8) as follows:

(7.9) 
$$\mathbf{v}_m = \sum_{s=1}^{\bar{s}} \sum_{r=1}^{r_s} \sum_{p=1}^{p_s^r} \beta_s^{r,p} \sum_{l=0}^{\min\{m, p-1\}} (-1)^m \binom{m}{l} (\mu_s)^{m-l} \mathbf{e}_s^{r,p-l},$$

where  $\beta_s^{r,p} \in C$  are arbitrary constants. One can evidently rewrite the formula (7.9) in the form

(7.10) 
$$\mathbf{v}_m = \sum_{s=1}^s \left(\mu_s\right)^m \mathbf{v}_{s,m},$$

where  $\mathbf{v}_{s,m} \in C^{n_s}$ ,  $s = 1, \ldots, \bar{s}$  are some functions of the argument m growing while m increases not faster than a polynomial of finite degree (namely, of degree  $n_s - 1$ ).

The formula (7.10) represents the whole space of the solutions to the homogeneous system (7.8). Assume (see the remark on page 1373) that if  $|\mu_s| = 1$  then  $r_s = n_s$ . In such a case the degree of the corresponding polynomial is zero. Then all the solutions of (7.8) which are bounded while  $m \longrightarrow \infty$  and only those solutions can be represented in the form

(7.11) 
$$\mathbf{v}_m^+ = \sum_{s: |\mu_s| \le 1} (\mu_s)^m \, \mathbf{v}_{s,m} \in C^+ \stackrel{\text{def}}{=} \bigoplus_{s: |\mu_s| \le 1} C^{n_s}.$$

By virtue of Lemma 7.3,  $\mathbf{v} \in C^+ \Leftrightarrow \mathbf{S}^+ \mathbf{v} = \mathbf{0}$ , where

(7.12) 
$$\mathbf{S}^+ = \prod_{s: |\mu_s| \le 1} \left( \mathbf{Q} - \mu_s \mathbf{I} \right)^{n_s}.$$

Further, one can also consider the system (7.8) for the negative m = 0, -1, -2, ...in the following form:

$$\mathbf{v}_{m-1} + \mathbf{Q}^{-1}\mathbf{v}_m = \mathbf{0}.$$

Eigenvalues of the matrix  $\mathbf{Q}^{-1}$  are obviously equal  $\mu_s^{-1}$ ,  $s = 1, \ldots, \bar{s}$ . Arguments analogous to ones presented above enable us to affirm that the subspace  $C^-$  of those solutions to (7.13) which vanish while  $m \longrightarrow -\infty$  is selected by the condition

(7.14) 
$$\mathbf{S}^{-\prime}\mathbf{v} = \mathbf{0} \iff \mathbf{v} \in C^{-},$$

where

(7.15) 
$$\mathbf{S}^{-\prime} = \prod_{s: |(\mu_s)^{-1}| < 1} \left( \mathbf{Q}^{-1} - (\mu_s)^{-1} \mathbf{I} \right)^{n_s}.$$

The subspace  $C^-$  is evidently not changed if one changes the operator  $\mathbf{S}^{-\prime}$  in (7.14) multiplying it by some nonsingular matrix as well as by some nonzero constant. Let  $n^- = \sum_{s: |(\mu_s)^{-1}| < 1} n_s$ . Putting  $\mathbf{S}^- = (-1)^{n^-} (\prod_{s: |(\mu_s)^{-1}| < 1} \mu_s) \mathbf{Q}^{n^-} \mathbf{S}^{-\prime}$  we obtain

(7.16) 
$$\mathbf{S}^{-} = \prod_{s: |\mu_s| > 1} \left( \mathbf{Q} - \mu_s \mathbf{I} \right)^{n_s}, \ \mathbf{S}^{-} \mathbf{v} = \mathbf{0} \ \Leftrightarrow \ \mathbf{v} \in C^{-}.$$

The operators  $\mathbf{S}^-$  (see (7.16)) and  $\mathbf{S}^+$  (see (7.12)) select the solutions of (7.13) and (7.8) according to their growth. The definitions of these operators coincide with the formulae (4.14). (Recall that the computation of matrix products in (4.14) is also realized with regard to the multiplicities of eigenvalues.) Furthermore, it is evident that the direct sum of subspaces of the solutions decreasing to the left and the solutions nonincreasing to the right yields the whole space of the solutions to the homogeneous system:  $C^n = C^+ \oplus C^-$  (taking into account the remark on page 1373).

Acknowledgment. We are very grateful to the reviewers for their useful comments.

## REFERENCES

- [1] D. GIVOLI, Non-reflecting boundary conditions, J. Comput. Phys., 94 (1991), pp. 1–29.
- [2] ——, Numerical Methods for Problems in Infinite Domains, Elsevier, Amsterdam, 1992.
- [3] S. V. TSYNKOV, An application of potential flow model for developing the artificial boundary conditions for the Euler equations. Part I, Preprint 40, Keldysh Institute of Applied Mathematics Russian Ac. Sci., Moscow, 1991. (In Russian.)
- [4] L. V. KANTOROVICH AND G. P. AKILOV, Functional Analysis, Nauka, Moscow, 1977. (In Russian.)
- [5] D. ANDERSON, J. TANNEHILL, AND R. PLETCHER, Computational Fluid Mechanics and Heat Transfer, Hemisphere Publishing Corporation, New York, 1984.
- [6] C. A. J. FLETCHER, Computational Techniques for Fluid Dynamics 2. Specific Techniques for Different Flow Categories, Springer-Verlag, Berlin, Heidelberg, 1988.
- [7] V. S. RYABEN'KII, Boundary equations with projections, Russian Math. Surveys, 40 (1985), pp. 147-183.
- [8] ——, Difference Potentials Method for Some Problems of Continuous Media Mechanics, Nauka, Moscow, 1987. (In Russian.)
- [9] A. P. CALDERON, Boundary-value problems for elliptic equations, in Proceedings of the Soviet-American Conference on Partial Differential Equations at Novosibirsk, Fizmatgiz, Moscow, 1963, pp. 303–304.
- [10] R. T. SEELEY, Singular integrals and boundary value problems, Amer. J. Math., 88 (1966), pp. 781-809.
- [11] I. L. SOFRONOV AND S. V. TSYNKOV, An application of potential flow model for developing the artificial boundary conditions for the Euler equations. Part II, Preprint 41, Keldysh Institute of Applied Mathematics Russian Ac. Sci., Moscow, 1991. (In Russian.)

- B. GUSTAFSSON AND A. SUNDSTRÖM, Incompletely parabolic problems in fluid dynamics, SIAM J. Appl. Math., 35 (1978), pp. 343-357.
- [13] J. NORDSTRÖM, The influence of open boundary conditions on the convergence to steady state for the Navier-Stokes equations, J. Comput. Phys., 85 (1989), pp. 210-244.
- [14] A. N. KOLMOGOROV AND S. V. FOMIN, Elements of the Theory of Functions and Functional Analysis, Nauka, Moscow, 1981. (In Russian.)
- [15] V. S. RYABEN'KII, Necessary and sufficient conditions for good definition of boundary value problems for systems of ordinary difference equations, U.S.S.R. Comput. Math. and Math. Phys., 4 (1964), pp. 43-61.
- [16] S.K. GODUNOV, A method of orthogonal successive substitution for the solution of systems of difference equations, U.S.S.R. Comput. Math. and Math. Phys., 2 (1963), pp. 1151-1165.
- [17] V. I. AGOSHKOV, Domain decomposition techniques for mathematical physics problems, in Computational Processes and Systems, Issue 8, Nauka, Moscow, 1991. (In Russian.)
- [18] A. JAMESON, W. SCHMIDT, AND E. TURKEL, Numerical solutions of the Euler equations by finite volume methods using Runge-Kutta time-stepping schemes, American Institute of Aeronautics and Astronautics Paper 81-1259, AIAA 14th Fluid and Plasma Dynamics Conference, Palo Alto, CA, 1981.
- [19] R. C. SWANSON AND E. TURKEL, A multistage time-stepping scheme for the Navier-Stokes equations, American Institute of Aeronautics and Astronautics paper 85-0035, AIAA 23rd Aerospace Science Meeting, Reno, NV, 1985.
- [20] ——, Artificial dissipation and central difference schemes for the Euler and Navier-Stokes equations, American Institute of Aeronautics and Astronautics paper 87-1107-CP, AIAA 8th Computational Fluid Dynamics Conference, Honolulu, HI, 1987.
- [21] S. V. TSYNKOV, An application of nonlocal external conditions to viscous flow computations, J. Comput. Phys., 16 (1995), pp. 212-225.