SIAM J. APPL. MATH.
Vol. 71, No. 4, pp. 1109–1122

QUASI-LACUNAE OF MAXWELL'S EQUATIONS*

S. V. PETROPAVLOVSKY† AND S. V. TSYNKOV‡

Abstract. Classical lacunae in the solutions of hyperbolic differential equations and systems (in the spaces of odd dimension) are a manifestation of the Huygens' principle. If the source terms are compactly supported in space and time, then, at any finite location in space, the solution becomes identically zero after a finite interval of time. In other words, the propagating waves have sharp aft fronts. For Maxwell's equations though, even if the currents that drive the field are compactly supported in time, they may still lead to the accumulation of charges. In that case, the solution won't have the lacunae per se. We show, however, that the notion of classical lacunae can be generalized, and that even when the steady-state charges are present, the waves still have sharp aft fronts. Yet behind those aft fronts, there is a nonzero electrostatic solution rather than one identically zero.

Key words. waves propagation, unsteadiness, Huygens' principle, odd-dimension spaces, aft fronts, accumulation of charge, electrostatic solution

AMS subject classifications. 35L05, 35Q61, 78A25

DOI. 10.1137/100798041

1. Introduction.

Consider the inhomogeneous scalar wave (d'Alembert) equation

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \Delta u = f(\boldsymbol{x}, t), \quad \boldsymbol{x} \in \mathbb{R}^3, \quad t \geqslant 0, \tag{1.1}$$

subject to zero initial conditions, and with the source term f that is compactly supported on a bounded domain $Q \subset \mathbb{R}^3 \times [0, +\infty)$. Then, the solution $u = u(\boldsymbol{x}, t)$ is known to have a secondary lacuna, or simply lacuna (see [19]), that we denote Λ:

$$u(\boldsymbol{x}, t) \equiv 0 \quad \forall(\boldsymbol{x}, t) \in \bigcap_{(\boldsymbol{\xi},\tau)\in Q} \left\{(\tilde{\boldsymbol{x}}, \tilde{t}) \big| |\tilde{\boldsymbol{x}} - \boldsymbol{\xi}| < c(\tilde{t} - \tau),\ \tilde{t} > \tau\right\} \stackrel{\text{def}}{=} \Lambda. \tag{1.2}$$

Mathematically, formula (1.2) implies that the lacuna Λ of the solution can be interpreted as the intersection of all characteristic cones (i.e., forward light cones) of the wave equation (1.1) once the vertex of the cone sweeps the support Q of the right-hand side $f(\boldsymbol{x}, t)$. From the standpoint of physics, Λ is the part of space-time on which the waves generated by a compactly supported source have already passed and for which the solution has become zero again. In contradistinction to that, the primary lacuna (as opposed to the secondary lacuna) is the part of space-time (ahead of the propagating fronts) where the waves have not reached yet. Hereafter, we will be focusing on the secondary lacunae, and will refer to those merely as lacunae.

The phenomenon of lacunae is inherently three-dimensional (more precisely, it pertains to the spaces of odd dimension). The surface of the lacuna represents the trajectory of aft (trailing) fronts of the waves. The existence of sharp aft fronts in

*Received by the editors June 9, 2010; accepted for publication (in revised form) April 19, 2011; published electronically July 19, 2011. This work was supported by the U.S. NSF, through grant DMS-0810963, and by the U.S. Air Force, through grants FA9550-07-1-0170 and FA9550-10-1-0092.
http://www.siam.org/journals/siap/71-4/79804.html

†Department of Applied Mathematics, Financial University under the Government of the Russian Federation, Moscow 125993, Russia (SPetropavlovsky@fa.ru).

‡Corresponding author. Department of Mathematics, North Carolina State University, Box 8205, Raleigh, NC 27695 (tsynkov@math.ncsu.edu, http://www.math.ncsu.edu/∼stsynkov).

odd-dimension spaces is known as the Huygens' principle, as opposed to the so-called wave diffusion which takes place in spaces of even dimension; see, e.g., [4, 27].

The question of identifying those hyperbolic equations and systems that admit diffusionless propagation of waves was first formulated by Hadamard [8, 9, 10]. He, however, did not know any examples other than the d'Alembert equation (1.1). The notion of lacunae was introduced and studied by Petrowsky in [19]; see also [4, Chapter VI]. He obtained conditions for the coefficients of hyperbolic equations that guaranteed the existence of lacunae. Subsequent developments can be found in the work by Atiyah, Bott, and Gårding in [1, 2]. However, since [19], no other constructive examples of either scalar equations or systems that satisfy the Huygens' principle have been found except for the wave equation (1.1) and its equivalents. Specifically, Matthisson [17] has shown that in the standard $3+1$-dimensional space-time with Minkowski metric, the only scalar hyperbolic equation that has lacunae is the wave equation (1.1). Later, Stellmacher [13, 22, 23] has built examples of nontrivial (i.e., irreducible to the wave equation) diffusionless equations, but only in the spaces $\mathbb{R}^n$ for odd $n \geqslant 5$. Günther and his school have provided examples of nontrivial diffusionless systems (as opposed to scalar equations) in the standard Minkowski $3+1$ space-time [3, 7, 21], and examples of nontrivial scalar Huygens' equations in a $3+1$-dimensional space-time but equipped with a different metric (the so-called plane wave metric that contains off-diagonal terms); see [3, 6, 7]. Lax and Phillips [16] have shown that the wave equation on the n-dimensional sphere, where $n \geqslant 3$ is odd, satisfies the Huygens' principle; this spherical wave equation can be transformed to the Euclidean wave equation locally, but not globally.

Maxwell's equations of electromagnetism [14],

$$\begin{aligned} &\frac{1}{c}\frac{\partial \boldsymbol{H}}{\partial t} + \mathrm{curl}\boldsymbol{E} = \boldsymbol{0}, \qquad && \mathrm{div}\boldsymbol{H} = 0, \\ &\frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} - \mathrm{curl}\boldsymbol{H} = -\frac{4\pi}{c}\boldsymbol{j}, \qquad && \mathrm{div}\boldsymbol{E} = 4\pi\rho, \end{aligned} \tag{1.3}$$

govern the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{H}$ that are driven by the electric current with density $\boldsymbol{j}$ and charge with density ρ. System (1.3) is a first order hyperbolic system (the Faraday law and the Ampère law) supplemented by two steady-state equations (the Gauss law for magnetism and the Gauss law for electricity). By taking divergence of the second unsteady equation of (1.3) and substituting the second steady-state equation, we obtain the continuity equation for the charges and currents:

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\boldsymbol{j} = 0. \tag{1.4}$$

From the standpoint of physics, (1.4) accounts for the conservation of electric charge, which is an independent property. The fact that (1.4) can also be derived from Maxwell's equations means that the conservation of charge is a necessary solvability condition for system (1.3).

By differentiating the first unsteady equation of (1.3) in time, substituting $\frac{\partial}{\partial t}\mathrm{curl}\boldsymbol{E}$ from the second unsteady equation, and employing the identity $\mathrm{curl}\,\mathrm{curl}\boldsymbol{H} = -\Delta\boldsymbol{H} + \mathrm{grad}\,\mathrm{div}\boldsymbol{H}$ along with the first steady-state equation of (1.3), we arrive at the inhomogeneous vector wave equation for $\boldsymbol{H}$:

$$\frac{1}{c^2}\frac{\partial^2 \boldsymbol{H}}{\partial t^2} - \Delta\boldsymbol{H} = \frac{4\pi}{c}\mathrm{curl}\boldsymbol{j}. \tag{1.5a}$$

A very similar argument yields the inhomogeneous vector wave equation for $\boldsymbol{E}$:

$$\frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} - \Delta \boldsymbol{E} = -4\pi\left[\frac{1}{c^2}\frac{\partial \boldsymbol{j}}{\partial t} + \text{grad}\rho\right]. \tag{1.5b}$$

In the Cartesian coordinates, the left-hand side of each equation (1.5a) or (1.5b) reduces to three independent scalar d'Alembert operators for individual field components (identical to the operator on the left-hand side of (1.1)). Therefore, according to the Matthisson criterion [17], one might expect that solutions to Maxwell's equations (1.3) will have lacunae. There is, however, a subtle issue related to the structure of the right-hand sides of (1.5a) and (1.5b).

Namely, even if the electric current $\boldsymbol{j} = \boldsymbol{j}(\boldsymbol{x}, t)$ is compactly supported in time, it may still cause the accumulation of charge $\rho = \rho(\boldsymbol{x}, t)$ according to the continuity equation (1.4). Hence, even after the current $\boldsymbol{j}$ ceases, a steady-state nonzero distribution of charge $\rho = \rho(\boldsymbol{x})$ may carry on in space for all subsequent moments of time. Consequently, the right-hand side of the wave equation (1.5b) will not, generally speaking, be compactly supported in space-time because of the term gradρ, and as such, solutions to Maxwell's equations (1.3) won't have the lacunae per se.

A particular case when Maxwell's equations still have lacunae is that of the solenoidal current, div$\boldsymbol{j} = 0$; see [20, 25]. If div$\boldsymbol{j} = 0$, then (1.4) implies $\frac{\partial \rho}{\partial t} = 0$, and provided that $\rho = 0$ at $t = 0$, the charge will remain zero thereafter. Hence, the right-hand side of (1.5b) will be compactly supported in space-time,[1] and the fields $\boldsymbol{H}$ and $\boldsymbol{E}$ will have lacunae. In general, however, this is not the case. Therefore, in this paper we introduce the notion of quasi-lacunae that generalizes the notion of classical lacunae for Maxwell's equations.

2. Quasi-lacunae. Let the electric current $\boldsymbol{j} = \boldsymbol{j}(\boldsymbol{x}, t)$ be compactly supported on the bounded region of space-time $Q \subset \mathbb{R}^3 \times [0, +\infty)$. With no loss of generality we can assume that $Q = Q_0 \times [0, T]$, where $Q_0 \subset \mathbb{R}^3$ is a bounded domain in space. Using Cartesian representation, we split the vector wave equation (1.5b) into three uncoupled scalar d'Alembert equations of type (1.1); in doing so, u can be any of the components E_x, E_y, or E_z, and f will be the respective component of the right-hand side of (1.5b). Solution of (1.1) subject to zero initial conditions is given by the Kirchhoff integral:

$$u(\boldsymbol{x}, t) = \frac{1}{4\pi} \iiint\limits_{|\boldsymbol{x}-\boldsymbol{\xi}| \leqslant ct} \frac{f(\boldsymbol{\xi}, t - |\boldsymbol{x} - \boldsymbol{\xi}|/c)}{|\boldsymbol{x} - \boldsymbol{\xi}|} d\boldsymbol{\xi}. \tag{2.1}$$

For subsequent analysis, it will be convenient to split the right-hand-side $f(\boldsymbol{x}, t)$ into the time-dependent and steady-state parts:

$$f(\boldsymbol{x}, t) = \begin{cases} f^{(1)}(\boldsymbol{x}, t), & \boldsymbol{x} \in Q_0, \quad 0 \leqslant t \leqslant T, \\ f^{(2)}(\boldsymbol{x}), & \boldsymbol{x} \in Q_0, \quad t > T, \end{cases} \tag{2.2}$$

where $f^{(1)}(\boldsymbol{x}, t)$ coincides with $-4\pi\left[\frac{1}{c^2}\frac{\partial j_x}{\partial t} + \frac{\partial \rho}{\partial x}\right]$, $-4\pi\left[\frac{1}{c^2}\frac{\partial j_y}{\partial t} + \frac{\partial \rho}{\partial y}\right]$, or $-4\pi\left[\frac{1}{c^2}\frac{\partial j_z}{\partial t} + \frac{\partial \rho}{\partial z}\right]$; $f^{(2)}(\boldsymbol{x})$ coincides with $-4\pi\frac{\partial \rho}{\partial x}$, $-4\pi\frac{\partial \rho}{\partial y}$, or $-4\pi\frac{\partial \rho}{\partial z}$; and $\rho = \rho(\boldsymbol{x}, t)$ is obtained by integrating the continuity equation (1.4) under the assumption that the initial charge

[1]The right-hand side of (1.5a) is compactly supported anyway.

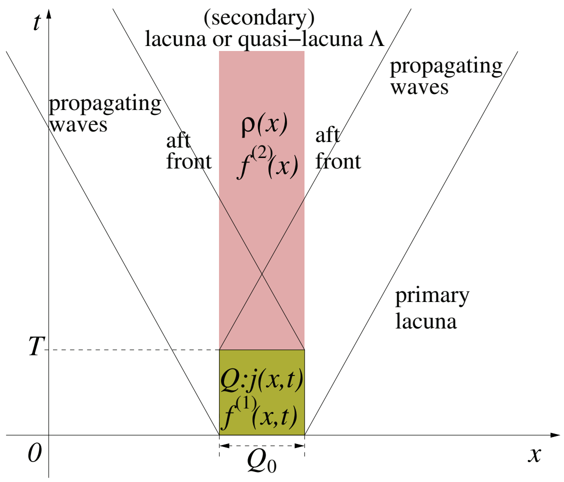

FIG. 2.1. *Schematic.*

is zero everywhere:

$$\rho(\boldsymbol{x},t)=\begin{cases}-\int_0^t \operatorname{div}\boldsymbol{j}(\boldsymbol{x},\tau)d\tau, & t\leqslant T,\\ -\int_0^T \operatorname{div}\boldsymbol{j}(\boldsymbol{x},\tau)d\tau, & t>T,\end{cases}\qquad \boldsymbol{x}\in Q_0. \tag{2.3}$$

From (2.3) is is clear that $\rho(\boldsymbol{x},t)\equiv\rho(\boldsymbol{x})$ for $t>T$. Accordingly, the Kirchhoff integral (2.1) also gets split into two parts: $u(\boldsymbol{x},t)=u^{(1)}(\boldsymbol{x},t)+u^{(2)}(\boldsymbol{x},t)$. The first part,

$$\begin{aligned} u^{(1)}(\boldsymbol{x},t) &= \frac{1}{4\pi}\iiint\limits_{|\boldsymbol{x}-\boldsymbol{\xi}|\leqslant ct}\frac{f^{(1)}(\boldsymbol{\xi},t-|\boldsymbol{x}-\boldsymbol{\xi}|/c)}{|\boldsymbol{x}-\boldsymbol{\xi}|}d\boldsymbol{\xi}\\ &= \frac{1}{4\pi}\iiint\limits_{c(t-T)\leqslant|\boldsymbol{x}-\boldsymbol{\xi}|\leqslant ct}\frac{f^{(1)}(\boldsymbol{\xi},t-|\boldsymbol{x}-\boldsymbol{\xi}|/c)}{|\boldsymbol{x}-\boldsymbol{\xi}|}d\boldsymbol{\xi}, \end{aligned} \tag{2.4}$$

is a solution to the d'Alembert equation (1.1) driven by the right-hand side $f^{(1)}(\boldsymbol{x},t)$, which is compactly supported in space-time on the domain $Q=Q_0\times[0,T]$; see (2.2). Note that the domain of integration in the second integral of formula (2.4) is defined by two inequalities; the left inequality appears there precisely because $f^{(1)}(\boldsymbol{x},t)$ switches off at $t=T$. If, for a given $(\boldsymbol{x},t)$, either of the inequalities $c(t-T)\leqslant|\boldsymbol{x}-\boldsymbol{\xi}|\leqslant ct$ in (2.4) is violated for all $\boldsymbol{\xi}\in O_0$, then the domain of integration is empty and hence $u^{(1)}(\boldsymbol{x},t)=0$. Violation of the right inequality corresponds to the area where the waves have not reached yet, i.e., to the primary lacuna ahead of the propagating front; see Figure 2.1. Violation of the left inequality corresponds to the secondary lacuna of $u^{(1)}(\boldsymbol{x},t)$ (see Figure 2.1), whose shape is given by (1.2).

The second part of (2.1), $u^{(2)}(\boldsymbol{x},t)$, can be considered driven by the right-hand side $\theta(t-T)f^{(2)}(\boldsymbol{x})$, where $\theta(\,\cdot\,)$ is the Heaviside function; see (2.2). This right-hand side is compactly supported in space but not in time; see Figure 2.1. As, however, its dependence on time is a mere step function, the substitution of this right-hand side into the Kirchhoff integral yields the following representation for $u^{(2)}$:

$$u^{(2)}(\boldsymbol{x},t) = \frac{1}{4\pi}\iiint\limits_{|\boldsymbol{x}-\boldsymbol{\xi}|\leqslant ct} \frac{\theta(t-|\boldsymbol{x}-\boldsymbol{\xi}|/c-T)f^{(2)}(\boldsymbol{\xi})}{|\boldsymbol{x}-\boldsymbol{\xi}|}d\boldsymbol{\xi}$$

$$= \frac{1}{4\pi}\iiint\limits_{|\boldsymbol{x}-\boldsymbol{\xi}|\leqslant c(t-T)} \frac{f^{(2)}(\boldsymbol{\xi})}{|\boldsymbol{x}-\boldsymbol{\xi}|}d\boldsymbol{\xi}. \tag{2.5}$$

The solution $u^{(2)}(\boldsymbol{x},t)$ depends on time only because the domain of integration in (2.5) depends on time: it is a ball of variable $\boldsymbol{\xi}$ centered at $\boldsymbol{x}$ with the radius $c(t-T)$. In particular, for any point $(\boldsymbol{x},t)$ inside the lacuna Λ defined by (1.2), the domain of integration in (2.5) contains $Q_0 = \operatorname{supp} f^{(2)}$. This, indeed, becomes apparent by setting $\tau = T$ in (1.2). Consequently, for the points inside the lacuna Λ, (2.5) can be recast as

$$u^{(2)}(\boldsymbol{x},t) = \frac{1}{4\pi}\iiint\limits_{\operatorname{supp} f^{(2)}} \frac{f^{(2)}(\boldsymbol{\xi})}{|\boldsymbol{x}-\boldsymbol{\xi}|}d\boldsymbol{\xi}, \quad (\boldsymbol{x},t)\in\Lambda. \tag{2.6}$$

Integral (2.6) is essentially the Newton's volume potential with compactly supported density $f^{(2)}(\boldsymbol{x})$; see [24, 27]. It is a solution to the Poisson equation $\Delta u^{(2)} = -f^{(2)}$. The only difference between expression (2.6) and the conventional Newton's potential is that representation (2.6) holds only inside the lacuna Λ given by (1.2), whereas the conventional potential is a function defined on the entire space $\mathbb{R}^3$ and vanishing at infinity. In this sense, for every given moment of time $t > T$ for which the lacuna Λ has already developed (see Figure 2.1), the solution $u^{(2)}(\boldsymbol{x},t)$ can be interpreted as the fragment of the true Newton's volume potential "cut out" by Λ.

If, on the other hand, the point $(\boldsymbol{x},t)$ is not inside the lacuna Λ, then there is either a partial overlap or no overlap at all between the integration region $|\boldsymbol{x}-\boldsymbol{\xi}| \leqslant c(t-T)$ in (2.5) and the domain $Q_0 = \operatorname{supp} f^{(2)}$. Indeed, the inequality in (1.2) becomes equality precisely at the aft front, and further into the region termed as propagating waves in Figure 2.1, the inequality no longer holds, which means that there will be points $\boldsymbol{\xi}\in Q_0$ such that $|\boldsymbol{x}-\boldsymbol{\xi}| > c(t-T)$. Moreover, if $(\boldsymbol{x},t)\notin\Lambda$ and the distance between $(\boldsymbol{x},t)$ and the aft front is greater than $\operatorname{diam} Q_0$, then $\forall\boldsymbol{\xi}\in Q_0 : |\boldsymbol{x}-\boldsymbol{\xi}| > c(t-T)$. It is also easy to see that the distance between the aft front and the front of the propagating wave is always greater than $\operatorname{diam} Q_0$ as long as $T > 0$. Consequently, the potential solution $u^{(2)}(\boldsymbol{x},t)$ that occupies the lacuna Λ dies off completely inside the propagating waves region before reaching the actual propagating front; see Figure 2.1.

The foregoing findings can be summarized as a theorem, which we formulate using specific electromagnetic variables as opposed to the generic notation.

THEOREM 2.1. *Let the electric and magnetic fields $\boldsymbol{E} = \boldsymbol{E}(\boldsymbol{x},t)$ and $\boldsymbol{H} = \boldsymbol{H}(\boldsymbol{x},t)$, $\boldsymbol{x}\in\mathbb{R}^3$, $t \geqslant 0$, be governed by Maxwell's equations* (1.3) *subject to zero initial conditions. Let the electric current $\boldsymbol{j} = \boldsymbol{j}(\boldsymbol{x},t)$ be compactly supported in space-time on the region $Q = Q_0\times[0,T]$, and let the charge $\rho = \rho(\boldsymbol{x},t)$ be equal to zero at $t = 0$ $\forall\boldsymbol{x}\in\mathbb{R}^3$. Then, the solution for the electric field $\boldsymbol{E}$ on the region of space-time Λ defined by* (1.2) *satisfies the electrostatic equation in space,*

$$\Delta\boldsymbol{E} = 4\pi\operatorname{grad}\rho \tag{2.7}$$

for every moment of time t for which $\Lambda \neq \emptyset$. The quantity gradρ *on the right-hand side of* (2.7) *is the gradient of the accumulated charge $\rho = \rho(\boldsymbol{x})$ given by* (2.3) *for*

$t > T$. The electric field on the bounded domain $\{\Lambda \cap \{t = \text{const}\}\} \subset \mathbb{R}^3$ is obtained by considering the solution $\boldsymbol{E}$ to the Poisson equation (2.7) *that vanishes at infinity and then truncating it from the entire $\mathbb{R}^3$ to Λ.*

For the magnetic field $\boldsymbol{H}$, the region Λ is a classical lacuna in the sense of Petrowsky; i.e., $\boldsymbol{H}(\boldsymbol{x}, t) = \boldsymbol{0}$ if $(\boldsymbol{x}, t) \in \Lambda$.

Proof. The implication for the electric field $\boldsymbol{E}$ is justified by the previous analysis, specifically, by the transformation of the Kirchhoff integral (2.5) into the Newton's potential (2.6). Indeed, it is the Newton's volume potential that yields the unique solution of the Poisson equation on $\mathbb{R}^3$ that vanishes at infinity. The result for the magnetic field $\boldsymbol{H}$ is obtained immediately, because the right-hand side of (1.5a) is compactly supported in space-time as long as the current $\boldsymbol{j}(\boldsymbol{x}, t)$ is compactly supported in space-time. ∎

Theorem 2.1 suggests that the propagating electromagnetic waves due to compactly supported electric currents in three dimensions still have sharp aft fronts, but behind those aft fronts there is, generally speaking, a nonzero electrostatic solution for the electric field. Hence, the notion of a classical lacuna for the electric field is replaced by a similar yet more general concept that we propose to call the quasi-lacuna. Note that in our previous work [26] we studied the residual field behind aft fronts due to the temporal dispersion of electric permittivity in the cold plasma [5, 18], and referred to the corresponding solutions as having weak lacunae. In contrast to [26], the phenomenon of quasi-lacunae introduced in this paper does not require that the propagation be dispersive.

The form of (2.7) also indicates that the electric field inside the quasi-lacuna Λ can be represented as $\boldsymbol{E} = -\text{grad}\varphi$, where the electrostatic potential φ satisfies the scalar Poisson equation

$$\Delta\varphi = -4\pi\rho$$

and also vanishes at infinity if considered on the entire $\mathbb{R}^3$.

3. Other forms of Maxwell's equations. Along with the classical Maxwell equations (1.3), one can consider their modified version that is made symmetric by adding a magnetic current to the Faraday law (on the right-hand side of the first unsteady equation of (1.3)) and magnetic charge to the Gauss law of magnetism (on the right-hand side of the first steady-state equation of (1.3)). The magnetic charges and currents are not physical [14, 15], but having them may sometimes be convenient from the standpoint of mathematical analysis; see, e.g., [25]. Applying the same arguments as in section 2 to the modified Maxwell equations, one can obtain that solutions for both the electric and magnetic field will now have quasi-lacunae as opposed to classical lacunae. In particular, there will be a nonzero magnetostatic field behind the propagating aft fronts.

It is also well known that, even though the Maxwell equations (1.3) are written with respect to two vector unknowns, $\boldsymbol{E}$ and $\boldsymbol{H}$, which altogether comprise six scalar quantities, the electromagnetic field is, in fact, fully determined by only four scalar quantities that can be taken in the form of the vector potential $\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{x}, t)$ and scalar potential $\varphi = \varphi(\boldsymbol{x}, t)$ so that

$$\boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} - \text{grad}\,\varphi \quad \text{and} \quad \boldsymbol{H} = \text{curl}\boldsymbol{A}. \tag{3.1}$$

The transformation of the potentials

$$\tilde{\boldsymbol{A}} = \boldsymbol{A} + \text{grad}\,\psi, \qquad \tilde{\varphi} = \varphi - \frac{1}{c}\frac{\partial\psi}{\partial t}, \tag{3.2}$$

where $\psi = \psi(\boldsymbol{x}, t)$ is an arbitrary scalar function of space and time, leaves the fields $\boldsymbol{E}$ and $\boldsymbol{H}$ given by (3.1) unchanged. Invariance of the fields with respect to the transformation (3.2) is known as the gauge invariance (see, e.g., [14, Chapter III]), and ψ is called the gauge function.

The gauge invariance allows one to impose additional constraints on $\boldsymbol{A}$ and φ. For example, the Lorenz[2] gauge [11, 12] is defined as follows:

$$\frac{1}{c}\frac{\partial \varphi}{\partial t} + \operatorname{div} \boldsymbol{A} = 0. \tag{3.3}$$

Substituting the first equation of (3.1) into the second unsteady equation of (1.3), the Ampère law, using the identity $\operatorname{curl}\operatorname{curl}\boldsymbol{A} = -\Delta \boldsymbol{A} + \operatorname{grad}\operatorname{div}\boldsymbol{A}$, and applying the Lorenz gauge (3.3), we arrive at the d'Alembert equation for $\boldsymbol{A}$:

$$\frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} - \Delta \boldsymbol{A} = \frac{4\pi}{c}\boldsymbol{j}. \tag{3.4}$$

As the electric current $\boldsymbol{j} = \boldsymbol{j}(\boldsymbol{x}, t)$ is compactly supported in space-time on the domain $Q = Q_0 \times [0, T] \subset \mathbb{R}^3 \times [0, +\infty)$ (see section 2), we conclude that the vector potential $\boldsymbol{A}$ under the Lorenz gauge (3.3) has a classical lacuna in the sense of Petrowsky.

Similarly, substituting the first equation of (3.1) into the second steady-state equation of (1.3), the Gauss law of electricity, and replacing the term $-\frac{1}{c}\frac{\partial}{\partial t}\operatorname{div} \boldsymbol{A}$ by $\frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2}$ through the Lorenz gauge (3.3), we obtain the d'Alembert equation for φ:

$$\frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} - \Delta \varphi = 4\pi\rho. \tag{3.5}$$

However, unlike in (3.4), the right-hand side of (3.5) is compactly supported in space (on the domain Q_0) but not necessarily in time (see Figure 2.1), because the electric charge may accumulate. Hence, the scalar potential φ under the Lorenz gauge (3.3) has a quasi-lacuna rather than a classical lacuna.

The Coulomb gauge [11] is defined as follows:

$$\operatorname{div} \boldsymbol{A} = 0. \tag{3.6}$$

Substituting the first equation of (3.1) into the second steady-state equation of (1.3) and using (3.6), we arrive at the scalar Poisson equation for φ:

$$\Delta \varphi = -4\pi\rho. \tag{3.7}$$

Unlike in (3.5), there is no variation in time for the solution φ to (3.7), except due to the varying charge density $\rho = \rho(\boldsymbol{x}, t)$, which becomes constant in time, $\rho = \rho(\boldsymbol{x})$, as of $t = T$; see Figure 2.1. Hence, under the Coulomb gauge the scalar potential φ is equivalent to the electrostatic potential defined on the entire $\mathbb{R}^3$.

The governing equation for the vector potential under the Coulomb gauge becomes

$$\frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} - \Delta \boldsymbol{A} = \frac{4\pi}{c}\boldsymbol{j} - \frac{1}{c}\frac{\partial}{\partial t}\operatorname{grad} \varphi. \tag{3.8}$$

The right-hand side of (3.8) is not compactly supported in space (although it is compactly supported in time on $[0, T]$). Therefore, neither $\boldsymbol{A}$ nor φ has either classical lacunae or quasi-lacunae under the Coulomb gauge (3.6).

[2]According to [12], the constraint (3.3) was introduced by L. V. Lorenz in the 1860s, but it is often misattributed to the better known H. A. Lorentz.

We note, however, that as the fields $\boldsymbol{E}$ and $\boldsymbol{H}$ are gauge invariant, their structure, including the existence and the shape of either lacunae or quasi-lacunae, is not affected by the changes that the potentials $\boldsymbol{A}$ and φ undergo under different gauges. Indeed, the governing equation (1.5a) for the magnetic field is obtained directly from (3.8) by applying the definition given by the second equation of (3.1). The governing equation (1.5b) for the electric field is recovered by applying the operation $-\frac{1}{c}\frac{\partial}{\partial t}$ to (3.8), taking the gradient of (3.7), subtracting the second equation from the first one, and using the definition of $\boldsymbol{E}$ given by the first equation of (3.1).

Finally, the gauge can also be taken as (see [14, Chapter III])

$$\varphi(\boldsymbol{x}, t) \equiv 0. \tag{3.9}$$

In fact, if $\rho(\boldsymbol{x}, t) \equiv 0$, then constraints (3.9) and (3.6) can be enforced simultaneously, but otherwise they cannot. Substituting the first equation of (3.1) along with (3.9) into the Gauss law of electricity (see (1.3)), we can write

$$\operatorname{div} \boldsymbol{A} = -4\pi c \int_0^t \rho d\tau.$$

Consequently, the governing equation for the vector potential becomes

$$\frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} - \Delta \boldsymbol{A} = \frac{4\pi}{c}\boldsymbol{j} + 4\pi c \int_0^t \operatorname{grad} \rho d\tau. \tag{3.10}$$

The right-hand side of (3.10) is compactly supported in space on the domain Q_0, but not necessarily in time. Moreover, unlike the right-hand side (2.2), it does not, generally speaking, become constant in time after a certain interval elapses. Therefore, the vector potential $\boldsymbol{A}$ has neither a classical lacuna nor a quasi-lacuna in the sense of section 2. We note, though, that as the integral on the right-hand side of (3.10) yields a mere linear growth for $t \geqslant T$, we can still, perhaps, talk about a quasi-lacuna, but such that the "steady-state" solution behind the aft fronts will be increasing linearly as a function of time. We also note that, similarly to the Coulomb gauge, the equations for the fields (1.5a) and (1.5b) can be recovered from (3.10) by applying the definitions $\boldsymbol{H} = \operatorname{curl} \boldsymbol{A}$ and $\boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}$, respectively.

4. Examples. To illustrate the previously introduced constructs, we first consider the case of spherical symmetry, for which all quantities may depend only on the radius r, i.e., $\frac{\partial}{\partial \theta} \equiv 0$ and $\frac{\partial}{\partial \phi} \equiv 0$, where (r, θ, ϕ) are the spherical coordinates. Let the charge density be given by

$$\rho(\boldsymbol{x}, t) = \theta(t)\delta(\boldsymbol{x}), \tag{4.1}$$

which is equivalent to a point charge (monopole) of unit magnitude that "pops up" in the origin at $t = 0$. The corresponding solution of the d'Alembert equation (3.5) can easily be obtained with the help of the Kirchhoff integral (2.1), which yields the following scalar electromagnetic potential under the Lorenz gauge:

$$\varphi(\boldsymbol{x}, t) = \frac{\theta(t - |\boldsymbol{x}|/c)}{|\boldsymbol{x}|}. \tag{4.2}$$

In the solution (4.2), the aft front coincides with the front $|\boldsymbol{x}| = ct$, and behind the front there is the Coulomb potential $|\boldsymbol{x}|^{-1}$. For the current density that corresponds to (4.1), we can write using the conservation (1.4):

$$\operatorname{div} \boldsymbol{j} = -\frac{\partial \rho}{\partial t} = -\delta(t)\delta(\boldsymbol{x}). \tag{4.3}$$

In the spherical coordinates, and under the assumption of spherical symmetry, (4.3) transforms into

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 j_r) = -\delta(t)\frac{\delta(r)}{4\pi r^2},$$

and integration yields the current $\boldsymbol{j} = (j_r, 0, 0)$ in the form

$$j_r(r,t) = -\frac{\delta(t)}{4\pi r^2}. \tag{4.4}$$

Under the same assumption (spherical symmetry), the vector d'Alembert equation (3.4) for $\boldsymbol{A} = (A_r, A_\theta, A_\phi)$, along with the source term given by (4.4), transforms into one scalar initial value problem for the radial component A_r,

$$\frac{1}{c^2}\frac{\partial^2 A_r}{\partial t^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial A_r}{\partial r}\right) - \frac{2A_r}{r^2},$$
$$A_r\big|_{t=0} = 0, \qquad \frac{\partial A_r}{\partial t}\bigg|_{t=0} = -\frac{1}{4\pi r^2},$$

whereas the solutions A_θ and A_ϕ to the two remaining scalar equations of (3.4) appear trivial. Hence, as for the current (4.4), the vector potential is radial, $\boldsymbol{A} = (A_r, 0, 0)$, and so the electric field $\boldsymbol{E}$ is obtained with the help of the first formula in (3.1). The magnetic field $\boldsymbol{H}$ in this case vanishes. We note, however, that as the current (4.4) is not compactly supported in space, then no secondary lacunae shall be expected to be observed in the solutions for either $\boldsymbol{A}$ or $\boldsymbol{E}$.

The noncompact nature of the current (4.4) can be understood and explained in the general perspective. Indeed, the current is a process of transporting the charge from one spatial location to another, and the very nature of conservation implies that if a charge appears at a given location ($\boldsymbol{x} = \boldsymbol{0}$ in the previous example) due to the operating current, then the same charge must be withdrawn from somewhere else. In other words, the charge of the same magnitude and opposite sign must appear elsewhere in the space. If we want to keep the current spherically symmetric and compactly supported at the same time, then the density of the resulting opposite charge that compensates (4.1) shall also be spherically symmetric,

$$\tilde{\rho}(\boldsymbol{x}, t) \equiv \tilde{\rho}(r,t), \qquad \int_0^\infty \tilde{\rho}(r,t) r^2 dr = -\frac{1}{4\pi} \quad \forall t > 0, \tag{4.5}$$

and compactly supported,

$$\tilde{\rho}(r,t) \equiv 0 \quad \text{for } r \geqslant R,$$

so that the integral in (4.5) can, in fact, be recast as

$$\int_0^R \tilde{\rho}(r,t) r^2 dr = -\frac{1}{4\pi} \quad \forall t > 0.$$

This, however, implies that the total charge inside the sphere of radius R centered at the origin is zero, and given that the distribution of charge is spherically symmetric, the corresponding electrostatic potential outside this sphere is zero as well. Hence, to have a nonzero electrostatic potential (such as (4.2)) that extends all the way to the propagating front $|\boldsymbol{x}| = ct$ for every $t > 0$, the charge (4.1) may not be drawn from a finite location, or equivalently, the opposite charge $\tilde{\rho}$ may not be compactly supported in space. That's why the current (4.4) draws the charge from infinity.

To restrict our consideration to compactly supported sources, we must therefore break away from spherical symmetry. As such, in our next example instead of the monopole (4.1) we consider the simplest nonsymmetric distribution of charge in the origin, which is a unit dipole that we choose aligned with the z axis:

$$\rho(\boldsymbol{x}, t) = -\theta(t)\delta'_z(\boldsymbol{x}). \tag{4.6}$$

According to the continuity equation (1.4), the current that leads to the charge distribution (4.6) is compactly supported in space and time:

$$\boldsymbol{j} = (j_x, j_y, j_z) = (0, 0, \delta(t)\delta(\boldsymbol{x})). \tag{4.7}$$

The electric field that corresponds to the charge (4.6) and current (4.7) can be obtained by solving (1.5b) with the help of the Kirchhoff integral or, equivalently, by means of convolution with the fundamental solution of the d'Alembert operator:

$$\mathcal{E}(\boldsymbol{x}, t) = \frac{1}{4\pi}\frac{\delta(t - |\boldsymbol{x}|/c)}{|\boldsymbol{x}|}. \tag{4.8}$$

The fundamental solution (4.8) is written for the scalar case, because individual Cartesian components of $\boldsymbol{E}$ can be computed independently. We have ($\boldsymbol{\xi} = (\xi, \eta, \zeta)$)

$$\begin{aligned} E_z(\boldsymbol{x}, t) = & -4\pi\mathcal{E} * \left[\frac{1}{c^2}\frac{\partial j_z}{\partial t} + \frac{\partial \rho}{\partial z}\right] \\ = & \iiint \frac{\delta(t - \tau - |\boldsymbol{x} - \boldsymbol{\xi}|/c)}{|\boldsymbol{x} - \boldsymbol{\xi}|}\left[-\frac{1}{c^2}\delta'(\tau)\delta(\boldsymbol{\xi}) + \theta(\tau)\delta''_{\zeta\zeta}(\boldsymbol{\xi})\right] d\tau d\boldsymbol{\xi} \\ = & \iiint \left[-\frac{1}{c^2}\frac{\delta'(t - |\boldsymbol{x} - \boldsymbol{\xi}|/c)}{|\boldsymbol{x} - \boldsymbol{\xi}|}\delta(\boldsymbol{\xi}) + \frac{\theta(t - |\boldsymbol{x} - \boldsymbol{\xi}|/c)}{|\boldsymbol{x} - \boldsymbol{\xi}|}\delta''_{\zeta\zeta}(\boldsymbol{\xi})\right] d\boldsymbol{\xi} \\ = & -\frac{1}{c^2}\frac{\delta'(t - |\boldsymbol{x}|/c)}{|\boldsymbol{x}|} + \frac{1}{c^2}\delta'\Big(t - \frac{|\boldsymbol{x}|}{c}\Big)\frac{z^2}{|\boldsymbol{x}|^3} \\ & -\frac{1}{c}\delta\Big(t - \frac{|\boldsymbol{x}|}{c}\Big)\left[\frac{1}{|\boldsymbol{x}|^2} - \frac{3z^2}{|\boldsymbol{x}|^4}\right] - \theta\Big(t - \frac{|\boldsymbol{x}|}{c}\Big)\left[\frac{1}{|\boldsymbol{x}|^3} - \frac{3z^2}{|\boldsymbol{x}|^5}\right]. \end{aligned} \tag{4.9}$$

The components E_x and E_y are obtained similarly, and altogether we arrive at

$$\begin{aligned} E_x = & \underbrace{\frac{1}{c^2}\delta'\Big(t - \frac{|\boldsymbol{x}|}{c}\Big)\frac{xz}{|\boldsymbol{x}|^3} + \frac{1}{c}\delta\Big(t - \frac{|\boldsymbol{x}|}{c}\Big)\frac{3xz}{|\boldsymbol{x}|^4}}_{\text{singular solution at the front}} + \underbrace{\theta\Big(t - \frac{|\boldsymbol{x}|}{c}\Big)\frac{3xz}{|\boldsymbol{x}|^5}}_{\substack{\text{electrostatic solution}\\ \text{behind the front}}}, \\ E_y = & \underbrace{\frac{1}{c^2}\delta'\Big(t - \frac{|\boldsymbol{x}|}{c}\Big)\frac{yz}{|\boldsymbol{x}|^3} + \frac{1}{c}\delta\Big(t - \frac{|\boldsymbol{x}|}{c}\Big)\frac{3yz}{|\boldsymbol{x}|^4}}_{\text{singular solution at the front}} + \underbrace{\theta\Big(t - \frac{|\boldsymbol{x}|}{c}\Big)\frac{3yz}{|\boldsymbol{x}|^5}}_{\substack{\text{electrostatic solution}\\ \text{behind the front}}}, \\ E_z = & \underbrace{\frac{1}{c^2}\delta'\Big(t - \frac{|\boldsymbol{x}|}{c}\Big)\left[\frac{z^2}{|\boldsymbol{x}|^3} - \frac{1}{|\boldsymbol{x}|}\right] - \frac{1}{c}\delta\Big(t - \frac{|\boldsymbol{x}|}{c}\Big)\left[\frac{1}{|\boldsymbol{x}|^2} - \frac{3z^2}{|\boldsymbol{x}|^4}\right]}_{\text{singular solution at the front}} \\ & \underbrace{-\theta\Big(t - \frac{|\boldsymbol{x}|}{c}\Big)\left[\frac{1}{|\boldsymbol{x}|^3} - \frac{3z^2}{|\boldsymbol{x}|^5}\right]}_{\substack{\text{electrostatic solution}\\ \text{behind the front}}}. \end{aligned} \tag{4.10}$$

As in the previous example, the aft front coincides with the leading front $|\boldsymbol{x}| = ct$ in the solution (4.10). The solution at the front is singular, whereas behind the front, i.e., inside the forward light cone, we have the electrostatic field from a point dipole. Hence, the electric field has a quasi-lacuna. The magnetic field can be obtained by solving (1.5a) with the right-hand side derived from (4.7), which yields

$$\begin{aligned} H_x &= -\frac{1}{c^2}\delta'\Big(t - \frac{|\boldsymbol{x}|}{c}\Big)\frac{y}{|\boldsymbol{x}|^2} - \frac{1}{c}\delta\Big(t - \frac{|\boldsymbol{x}|}{c}\Big)\frac{y}{|\boldsymbol{x}|^3},\\ H_y &= \frac{1}{c^2}\delta'\Big(t - \frac{|\boldsymbol{x}|}{c}\Big)\frac{x}{|\boldsymbol{x}|^2} + \frac{1}{c}\delta\Big(t - \frac{|\boldsymbol{x}|}{c}\Big)\frac{x}{|\boldsymbol{x}|^3},\\ H_z &= 0. \end{aligned} \tag{4.11}$$

The solution (4.11) is nonzero only at the front (it is singular there), whereas behind the front it is identically equal to zero. Hence, the magnetic field has a conventional lacuna, as suggested by Theorem 2.1.

To remove the singular behavior, and to consider solutions for which the aft front and the leading front would not necessarily coincide, let us consider the classical hat function from the theory of distributions:

$$\omega_\epsilon(t) = \begin{cases} C_\epsilon e^{-\frac{\epsilon^2}{\epsilon^2 - t^2}}, & |t| \leqslant \epsilon,\\ 0, & |t| > \epsilon. \end{cases} \tag{4.12}$$

The constant C_ϵ in (4.12) is chosen so that $\int \omega_\epsilon(t)dt = 1$. The function $\omega_\epsilon(t)$ defined by (4.12) is compactly supported on the interval $-\epsilon \leqslant t \leqslant \epsilon$, and it is known that $\omega_\epsilon(t) \to \delta(t)$ as $\epsilon \to +0$ in the sense of distributions $\mathcal{D}'$. The same is true for all the respective derivatives of ω_ϵ and δ; see [27]. We also introduce the primitive of $\omega_\epsilon(t)$,

$$\Omega_\epsilon(t) = \int_{-\infty}^{t} \omega_\epsilon(\tau)d\tau, \tag{4.13}$$

for which the following convergence takes place: $\Omega_\epsilon(t) \to \theta(t)$ as $\epsilon \to +0$ in $\mathcal{D}'$.

Instead of the source terms (4.6) and (4.7) that have a singular behavior in time, let us now consider the charge and the current defined with the help of the functions (4.13) and (4.12):

$$\rho(\boldsymbol{x}, t) = -\Omega_\epsilon(t)\delta'_z(\boldsymbol{x}) \tag{4.14}$$

and

$$\boldsymbol{j} = (j_x, j_y, j_z) = (0, 0, \omega_\epsilon(t)\delta(\boldsymbol{x})). \tag{4.15}$$

Both ρ of (4.14) and $\boldsymbol{j}$ of (4.15) are smooth functions of time, $\boldsymbol{j}$ is also compactly supported on $[-\epsilon, \epsilon]$, and as $\epsilon \to +0$, they converge to ρ of (4.6) and $\boldsymbol{j}$ of (4.7), respectively, in the sense of $\mathcal{D}'$. In accordance with the constructs of section 2 (see (2.2)), we split the right-hand side of (1.5b) into two parts:

$$\begin{aligned} \boldsymbol{f}^{(1)}(\boldsymbol{x}, t) &= -4\pi\left[\frac{1}{c^2}\frac{\partial \boldsymbol{j}}{\partial t} + \text{grad}\rho\right]\theta(\epsilon - t)\\ &= -\frac{4\pi}{c^2}\frac{\partial \boldsymbol{j}}{\partial t} - 4\pi\,\text{grad}\rho\,\theta(\epsilon - t) \end{aligned} \tag{4.16a}$$

and

$$\boldsymbol{f}^{(2)}(\boldsymbol{x},t) = -4\pi\,\mathrm{grad}\rho\,\theta(t-\epsilon), \tag{4.16b}$$

where ρ and $\boldsymbol{j}$ are given by (4.14) and (4.15), respectively. Note that $\theta(\epsilon - t)$ on the right-hand side of (4.16a) applies only to the gradρ term, because $\frac{\partial \boldsymbol{j}}{\partial t}$ is compactly supported on $[-\epsilon,\epsilon]$; see (4.15). Computing the convolution of the partial right-hand sides (4.16a) and (4.16b) with the fundamental solution $\mathcal{E}$ of (4.8), we can write for the z component of the electric field (cf. (4.9))

$$\begin{aligned}
E_z^{(1)}(\boldsymbol{x},t) &= \mathcal{E} * f_z^{(1)} = -4\pi\mathcal{E} * \left[\frac{1}{c^2}\frac{\partial j_z}{\partial t} + \frac{\partial\rho}{\partial z}\theta(\epsilon - t)\right]\\
&= \iiint \frac{\delta(t-\tau-|\boldsymbol{x}-\boldsymbol{\xi}|/c)}{|\boldsymbol{x}-\boldsymbol{\xi}|}\left[-\frac{1}{c^2}\omega_\epsilon'(\tau)\delta(\boldsymbol{\xi}) + \Omega_\epsilon(\tau)\theta(\epsilon-\tau)\delta''_{\zeta\zeta}(\boldsymbol{\xi})\right]d\tau d\boldsymbol{\xi}\\
&= \iiint \Bigg[-\frac{1}{c^2}\frac{\omega_\epsilon'(t-|\boldsymbol{x}-\boldsymbol{\xi}|/c)}{|\boldsymbol{x}-\boldsymbol{\xi}|}\delta(\boldsymbol{\xi})\\
&\qquad + \frac{\Omega_\epsilon(t-|\boldsymbol{x}-\boldsymbol{\xi}|/c)}{|\boldsymbol{x}-\boldsymbol{\xi}|}\theta\Big(\epsilon - \Big(t - \frac{|\boldsymbol{x}-\boldsymbol{\xi}|}{c}\Big)\Big)\delta''_{\zeta\zeta}(\boldsymbol{\xi})\Bigg]d\boldsymbol{\xi}
\end{aligned} \tag{4.17a}$$

and

$$\begin{aligned}
E_z^{(2)}(\boldsymbol{x},t) &= \mathcal{E} * f_z^{(2)} = -4\pi\mathcal{E} * \frac{\partial\rho}{\partial z}\theta(t-\epsilon)\\
&= \iiint \frac{\delta(t-\tau-|\boldsymbol{x}-\boldsymbol{\xi}|/c)}{|\boldsymbol{x}-\boldsymbol{\xi}|}\left[\Omega_\epsilon(\tau)\theta(\tau-\epsilon)\delta''_{\zeta\zeta}(\boldsymbol{\xi})\right]d\tau d\boldsymbol{\xi}\\
&= \iiint \left[\frac{\Omega_\epsilon(t-|\boldsymbol{x}-\boldsymbol{\xi}|/c)}{|\boldsymbol{x}-\boldsymbol{\xi}|}\theta\Big(\Big(t-\frac{|\boldsymbol{x}-\boldsymbol{\xi}|}{c}\Big)-\epsilon\Big)\delta''_{\zeta\zeta}(\boldsymbol{\xi})\right]d\boldsymbol{\xi}.
\end{aligned} \tag{4.17b}$$

Adding (4.17a) and (4.17b) together, we have (cf. (4.9))

$$\begin{aligned}
E_z(\boldsymbol{x},t) &= E_z^{(1)}(\boldsymbol{x},t) + E_z^{(2)}(\boldsymbol{x},t)\\
&= \iiint \left[-\frac{1}{c^2}\frac{\omega_\epsilon'(t-|\boldsymbol{x}-\boldsymbol{\xi}|/c)}{|\boldsymbol{x}-\boldsymbol{\xi}|}\delta(\boldsymbol{\xi}) + \frac{\Omega_\epsilon(t-|\boldsymbol{x}-\boldsymbol{\xi}|/c)}{|\boldsymbol{x}-\boldsymbol{\xi}|}\delta''_{\zeta\zeta}(\boldsymbol{\xi})\right]d\boldsymbol{\xi}\\
&= -\frac{1}{c^2}\frac{\omega_\epsilon'(t-|\boldsymbol{x}|/c)}{|\boldsymbol{x}|} + \frac{1}{c^2}\omega_\epsilon'\Big(t-\frac{|\boldsymbol{x}|}{c}\Big)\frac{z^2}{|\boldsymbol{x}|^3}\\
&\qquad - \frac{1}{c}\omega_\epsilon\Big(t-\frac{|\boldsymbol{x}|}{c}\Big)\left[\frac{1}{|\boldsymbol{x}|^2} - \frac{3z^2}{|\boldsymbol{x}|^4}\right] - \Omega_\epsilon\Big(t-\frac{|\boldsymbol{x}|}{c}\Big)\left[\frac{1}{|\boldsymbol{x}|^3} - \frac{3z^2}{|\boldsymbol{x}|^5}\right],
\end{aligned} \tag{4.18}$$

and the overall solution for the electric field becomes (cf. (4.10))

$$\begin{aligned}
E_x &= \underbrace{\frac{1}{c^2}\omega_\epsilon'\Big(t-\frac{|\boldsymbol{x}|}{c}\Big)\frac{xz}{|\boldsymbol{x}|^3} + \frac{1}{c}\omega_\epsilon\Big(t-\frac{|\boldsymbol{x}|}{c}\Big)\frac{3xz}{|\boldsymbol{x}|^4}}_{\text{propagating waves}} + \underbrace{\Omega_\epsilon\Big(t-\frac{|\boldsymbol{x}|}{c}\Big)\frac{3xz}{|\boldsymbol{x}|^5}}_{\substack{\text{electrostatic solution}\\ \text{that gradually dies off}}},\\
E_y &= \underbrace{\frac{1}{c^2}\omega_\epsilon'\Big(t-\frac{|\boldsymbol{x}|}{c}\Big)\frac{yz}{|\boldsymbol{x}|^3} + \frac{1}{c}\omega_\epsilon\Big(t-\frac{|\boldsymbol{x}|}{c}\Big)\frac{3yz}{|\boldsymbol{x}|^4}}_{\text{propagating waves}} + \underbrace{\Omega_\epsilon\Big(t-\frac{|\boldsymbol{x}|}{c}\Big)\frac{3yz}{|\boldsymbol{x}|^5}}_{\substack{\text{electrostatic solution}\\ \text{that gradually dies off}}},\\
E_z &= \underbrace{\frac{1}{c^2}\omega_\epsilon'\Big(t-\frac{|\boldsymbol{x}|}{c}\Big)\left[\frac{z^2}{|\boldsymbol{x}|^3} - \frac{1}{|\boldsymbol{x}|}\right] - \frac{1}{c}\omega_\epsilon\Big(t-\frac{|\boldsymbol{x}|}{c}\Big)\left[\frac{1}{|\boldsymbol{x}|^2} - \frac{3z^2}{|\boldsymbol{x}|^4}\right]}_{\text{propagating waves}}
\end{aligned}$$

$$-\underbrace{\Omega_\epsilon\Big(t-\frac{|\boldsymbol{x}|}{c}\Big)\left[\frac{1}{|\boldsymbol{x}|^3}-\frac{3z^2}{|\boldsymbol{x}|^5}\right]}_{\substack{\text{electrostatic solution}\\ \text{that gradually dies off}}}. \tag{4.19}$$

Unlike the solution (4.10), for which the leading front and the aft front coincide, solution (4.19) has a finite size propagating waves region between the leading front $|\boldsymbol{x}| = c(t+\epsilon)$ and the aft front $|\boldsymbol{x}| = c(t-\epsilon)$. The electrostatic solution behind the aft front in (4.19) is the same dipole field as in (4.10), but unlike in (4.10), where it is cut off abruptly at the front, in (4.19) the electrostatic solution gradually dies off in the region $c(t-\epsilon) \leqslant |\boldsymbol{x}| \leqslant c(t+\epsilon)$. Of course, as $\epsilon \to +0$, the solution (4.19) converges to the solution (4.10) in the sense of distributions $\mathcal{D}'$.

Let us also emphasize that, as this last example shows, the notion of a front (specifically, a sharp front) does not necessarily imply a singularity of any kind. Indeed, the solution (4.19) is smooth at both the leading front and the aft front because the functions (4.12) and (4.13) are smooth. Hence, the fronts can rather be described as clearly identifiable surfaces at which the behavior of the solution changes in a particular way; for example, the solution becomes purely electrostatic.

5. Discussion. We have shown that electromagnetic waves driven by compactly supported electric currents in three dimensions always have sharp aft fronts. However, unlike in the case of the classical Huygens' principle, the solution behind these aft fronts is not always zero. Specifically, if the steady-state electric charge accumulates in space after the current ceases, then there will be a nonzero electrostatic field behind aft fronts of the propagating waves, i.e., on the region that in the case of scalar propagation would have been a lacuna of the solution in the sense of Petrowsky. Accordingly, we propose to call such regions quasi-lacunae as opposed to the conventional lacunae.

As the unsteady electromagnetic waves propagate away from their sources (compactly supported electric currents), the quasi-lacuna is expanding; see Figure 2.1. In other the words, the region of space occupied by the electrostatic field increases in size as time elapses. Eventually, as the time t approaches infinity, the electrostatic solution will be present on the entire space $\mathbb{R}^3$. This evolution process can be interpreted as gradual onset of the steady-state electrostatic solution.

We have also analyzed some alternative forms of Maxwell's equations and shown, in particular, that the vector and scalar potentials may or may not have lacunae or quasi-lacunae under different gauges. As, however, the electric and magnetic fields themselves are gauge invariant, our conclusions regarding the presence and the shape of their lacunae or quasi-lacunae remain unaffected by the choice of a specific gauge.

Acknowledgment. We are very thankful to the anonymous referee for his/her useful and encouraging comments that have undoubtedly helped us improve the paper. In particular, the idea of analyzing different gauges for Maxwell's equations (see section 3) was proposed by the referee, and the examples in section 4 were also motivated by his/her suggestions.

REFERENCES

[1] M. F. Atiyah, R. Bott, and L. Gårding, *Lacunas for hyperbolic differential operators with constant coefficients.* I, Acta Math., 124 (1970), pp. 109–189.

[2] M. F. Atiyah, R. Bott, and L. Gårding, *Lacunas for hyperbolic differential operators with constant coefficients.* II, Acta Math., 131 (1973), pp. 145–206.

[3] M. Belger, R. Schimming, and V. Wünsch, *A survey on Huygens' principle*, Z. Anal. Anwendungen, 16 (1997), pp. 9–36.
[4] R. Courant and D. Hilbert, *Methods of Mathematical Physics. Volume* II, Wiley, New York, 1962.
[5] V. L. Ginzburg, *The Propagation of Electromagnetic Waves in Plasmas*, Internat. Ser. Monogr. Electromagnetic Waves 7, Pergamon Press, Oxford, UK, 1964.
[6] P. Günther, *Ein Beispiel einer nichttrivialen Huygensschen Differentialgleichung mit vier unabhängigen Variablen*, Arch. Ration. Mech. Anal., 18 (1965), pp. 103–106 (in German).
[7] P. Günther, *Huygens' Principle and Hyperbolic Equations* (with appendices by V. Wünsch), Perspect. Math. 5, Academic Press, Boston, MA, 1988.
[8] J. Hadamard, *Lectures on Cauchy's Problem in Linear Partial Differential Equations*, Yale University Press, New Haven, CT, 1923.
[9] J. Hadamard, *Problème de Cauchy*, Hermann, Paris, 1932 (in French).
[10] J. Hadamard, *The problem of diffusion of waves*, Ann. of Math. (2), 43 (1942), pp. 510–522.
[11] J. D. Jackson, *From Lorenz to Coulomb and other explicit gauge transformations*, Amer. J. Phys., 70 (2002), pp. 917–928.
[12] J. D. Jackson and L. B. Okun, *Historical roots of gauge invariance*, Rev. Modern Phys., 73 (2001), pp. 663–680.
[13] J. E. Lagnese and K. L. Stellmacher, *A method of generating classes of Huygens' operators*, J. Math. Mech., 17 (1967), pp. 461–472.
[14] L. D. Landau and E. M. Lifshitz, *Course of Theoretical Physics, Vol.* 2*, The Classical Theory of Fields* (translated from the Russian by M. Hamermesh), 4th ed., Pergamon Press, Oxford, UK, 1975.
[15] L. D. Landau and E. M. Lifshitz, *Course of Theoretical Physics. Vol.* 8*, Electrodynamics of Continuous Media* (translated from the 2nd Russian ed. by J. B. Sykes, J. S. Bell, and M. J. Kearsley), Pergamon International Library of Science, Technology, Engineering and Social Studies, Pergamon Press, Oxford, UK, 1984.
[16] P. D. Lax and R. S. Phillips, *An example of Huygens' principle*, Comm. Pure Appl. Math., 31 (1978), pp. 415–421.
[17] M. Matthisson, *Le probléme de Hadamard relatif á la diffusion des ondes*, Acta Math., 71 (1939), pp. 249–282 (in French).
[18] D. B. Melrose and R. C. McPhedran, *Electromagnetic Processes in Dispersive Media. A Treatment Based on the Dielectric Tensor*, Cambridge University Press, Cambridge, UK, 1991.
[19] I. Petrowsky, *On the diffusion of waves and the lacunas for hyperbolic equations*, Matematicheskii Sbornik (Recueil Mathématique), 17 (1945), pp. 289–370.
[20] H. Qasimov and S. Tsynkov, *Lacunae based stabilization of PMLs*, J. Comput. Phys., 227 (2008), pp. 7322–7345.
[21] R. Schimming, *A review of Huygens' principle for linear hyperbolic differential equations*, in Proceedings of the IMU Symposium "Group-Theoretical Methods in Mechanics" (Novosibirsk, USSR, 1978), USSR Acad. Sci., Siberian Branch, Novosibirsk, Russia, 1978, pp. 214–225.
[22] K. L. Stellmacher, *Ein Beispiel einer Huyghensschen Differentialgleichung*, Nachr. Akad. Wiss. Göttingen. Math. Phys. Kl. Math.-Phys. Chem. Abt., 1953 (1953), pp. 133–138 (in German).
[23] K. L. Stellmacher, *Eine Klasse huyghenscher Differentialgleichungen und ihre Integration*, Math. Ann., 130 (1955), pp. 219–233 (in German).
[24] A. N. Tikhonov and A. A. Samarskii, *Equations of Mathematical Physics*, Pergamon Press, Oxford, UK, 1963.
[25] S. V. Tsynkov, *On the application of lacunae-based methods to Maxwell's equations*, J. Comput. Phys., 199 (2004), pp. 126–149.
[26] S. V. Tsynkov, *Weak lacunae of electromagnetic waves in dilute plasma*, SIAM J. Appl. Math., 67 (2007), pp. 1548–1581.
[27] V. S. Vladimirov, *Equations of Mathematical Physics*, Dekker, New York, 1971.