



Active control of sound with variable degree of cancellation

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ABSTRACT

We formulate and solve a control problem for the field (e.g., time-harmonic sound) governed by a linear PDE or system on a composite domain in \mathbb{R}^n . Namely, we require that simultaneously and independently on each subdomain the sound generated in its complement be attenuated to a desired degree. This goal is achieved by adding special control sources defined only at the interface between the subdomains. We present a general solution for controls in the continuous and discrete setting.

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1. Introduction

Active control of sound is a way to attain a desirable alteration of the acoustic field by means of modifying the sources or adding new sources. This area has been studied for the past four decades, see, e.g., [1–6]; in general, active control of sound is an inverse source problem [7].

In acoustics, the active control (AC) problem is often identified with active shielding (AS) problem [8], in which a given subdomain needs to be shielded from the noise generated outside. Shielding can be achieved by introducing the control sources on the perimeter of the shielded domain. The analysis in the literature is usually limited to considering only external sources of noise and overall unbounded regions. The use of Calderon's boundary projections and the method of difference potentials [9], [10, Chapter 14] allows us to take into account the effect of both internal sources and external boundaries. Previously, we have obtained a general solution of the AS problem for second order equations in both continuous [11] and finite-difference formulation [12]. Our approach requires minimum a priori information—only the knowledge of the overall solution (total acoustic field) on the boundary of the protected region. It does not require any information on either actual form of the noise sources or properties of the medium. In [13], the technique has been generalized to obtain the continuous and discrete solution of the problem in the form of surface controls. Consistency of the discrete and continuous solutions has been shown in [14,15]. The problem of selective shielding in composite regions has been formulated and solved in [16,17] for the discrete and continuous formulation, respectively. In [18], it has been shown how to take into account the feedback of active control sources on the input data.

Hereafter, we analyze the composite AC problem, in which the desired extent of cancellation or amplification of sound on each of the two subdomains can be prescribed. In doing so, the complete shielding of individual subdomains (whether a given one or both from one another) is attained in the special limit cases. Similarly to the conventional AS, solution of the new AC problem requires no knowledge of either the sources of noise or the boundary conditions. However, unlike previously, to attain a predetermined (non-total) degree of cancellation, one additionally needs to know the contribution

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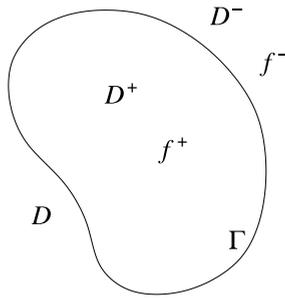


Fig. 1. Schematic.

from one of the two sides to the overall field at the interface. The limit cases of total cancellation (complete shielding) can still be done based on the knowledge of the overall field only.

2. Continuous formulation of the problem

Consider the following linear boundary value problem (BVP) on the domain $D \subseteq \mathbb{R}^n$:

$$\begin{aligned} Lu = f, \quad \mathbf{x} \in D, \\ u \in \mathcal{E}_D. \end{aligned} \tag{1}$$

Here, L is a linear differential operator and \mathcal{E}_D is a function space that guarantees the solvability and uniqueness, provided that the right-hand side f belongs to another appropriate space F_D , e.g., $F_D = \mathcal{D}'_D$. The quantity u in (1) may be interpreted as acoustic pressure, in which case L is the Helmholtz operator (see Section 3); u may also be a vector field with pressure and velocity as components. For simplicity, u will hereafter be referred to as sound.

We assume that the definition of \mathcal{E}_D includes the boundary conditions on ∂D that may be inherited from physics, say, sound-soft, sound-hard, or impedance boundary conditions at a finite boundary or Sommerfeld radiation conditions at infinity.¹ We also assume that the original physical problem (which is formulated using regular functions rather than distributions) may only be weakly sensitive to perturbations of the data.

The function u is said to be a generalized solution of BVP (1) if $\forall \phi(\bar{D}) \in C_0^\infty(\bar{D}) : \langle Lu, \phi \rangle = \langle f, \phi \rangle$. Here, $\langle f, \phi \rangle$ denotes a linear continuous functional associated with the given generalized function (distribution) f .

Let us also introduce a domain $D^+ \subset D$ and its complement $D^- = D \setminus D^+$, where $\Gamma = \partial D^+$ is sufficiently smooth (note, $\Gamma \subset D^-$), see Fig. 1. We require that if $f \in F_D$, then $\theta_{D^+}f \in F_D$, where θ_{D^+} is the indicator equal to 1 on D^+ and equal to 0 on D^- . Along with (1), consider a similar BVP:

$$\begin{aligned} Lv = f + g, \quad \mathbf{x} \in D, \\ v \in \mathcal{E}_D. \end{aligned} \tag{2}$$

The function g , $\text{supp } g \subset D^-$, is called an active control if the solution of (2) satisfies some predetermined constraints on D^+ . For example, v may be required to coincide with the portion of the overall field due only to the sources located inside D^+ , which is the complete shielding of D^+ .

Hereafter, we will assume that solutions of BVPs (1) and (2) are known at the interface Γ either from measurements or from computations; e.g., acoustic pressure can be measured by microphones.

Let $f^+ \stackrel{\text{def}}{=} \theta_{D^+}f$ and $f^- \stackrel{\text{def}}{=} \theta_{D^-}f$ so that $f = f^+ + f^-$, and consider the following two BVPs:

$$\begin{aligned} Lu^+ = f^+, \quad \mathbf{x} \in D, \\ u^+ \in \mathcal{E}_D, \end{aligned} \tag{3}$$

and

$$\begin{aligned} Lu^- = f^-, \quad \mathbf{x} \in D, \\ u^- \in \mathcal{E}_D. \end{aligned} \tag{4}$$

From the linearity of the BVPs (1), (3) and (4), it immediately follows that $u = u^+ + u^-$, $\mathbf{x} \in D$.

Next, we introduce the control function $g = g_+$:

$$g_+ = -\theta_{D^-}L(\theta_{D^+}u). \tag{5}$$

The distribution $L(\theta_{D^+}u)$ in (5) can be represented as follows [18]:

$$L(\theta_{D^+}u) = L_\Gamma(u) + \theta_{D^+}\{Lu\}, \tag{6}$$

¹ In the case of classical solutions, the definition of \mathcal{E}_D may also include the desired extent of regularity for u .

where $\{Lu\}$ denotes the regular part of Lu , and the singular distribution $L_\Gamma(u)$ is fully determined by the Cauchy data of the function u on the boundary Γ [18] (also see example in Section 3):

$$\text{Tr}_\Gamma u = \left(u, \frac{\partial u}{\partial \mathbf{n}}, \dots, \frac{\partial^{k-1} u}{\partial \mathbf{n}^{k-1}} \right)_\Gamma^T. \tag{7}$$

In formula (7), k is the order of the operator L and \mathbf{n} is the outward normal to the boundary Γ .

From (5) and (6), it is clear that $\text{supp}g_+ = \Gamma$ and g_+ is fully determined by $\text{Tr}_\Gamma u$ of (7), which is assumed known. Hence, we arrive at the following theorem.

Theorem 1. *The control g_+ of (5) renders complete shielding of D^+ from the sound generated in D^- . The field due to both the primary source f and secondary control source $g = g_+$ is given by:*

$$v = \begin{cases} u^+, & \text{if } \mathbf{x} \in D^+, \\ u + u^+, & \text{if } \mathbf{x} \in D^-. \end{cases}$$

Proof. Consider

$$f + g = Lu - L(\theta_{D^+}u) + \theta_{D^+}L(\theta_{D^+}u) = L(\theta_{D^-}u) + f^+.$$

Then, the solution of BVP (2) is given by

$$v = \theta_{D^-}u + u^+.$$

This solution is unique, and clearly, $v_{D^+} = u_{D^+}^+$. \square

Thus, the domain D^+ appears shielded from the sources f^- , whereas on the domain D^- the overall field gets incremented by u^+ due to the secondary sources g_+ .

Next, let us introduce the following control functions:

$$g^+ = \theta_{D^+}L(\theta_{D^-}u^+) \quad \text{and} \quad g^- = \theta_{D^-}L(\theta_{D^+}u^-), \tag{8}$$

as well as their linear combination:

$$g_\alpha(\alpha^+, \alpha^-) = \alpha^+g^+ + \alpha^-g^-. \tag{9}$$

Since $\{g^+\} = 0$ and $\{g^-\} = 0$, we have $\text{supp}g_\alpha(\alpha^+, \alpha^-) \subset \Gamma$, and the following theorem holds.

Theorem 2. *The field due to both the primary source f and the control source $g = g_\alpha$ of (9) is*

$$v = \begin{cases} u^+ + (1 + \alpha^-)u^-, & \text{if } \mathbf{x} \in D^+, \\ u^- + (1 + \alpha^+)u^+, & \text{if } \mathbf{x} \in D^-. \end{cases} \tag{10}$$

Proof. One can see that

$$\begin{aligned} f + \alpha^+g^+ + \alpha^-g^- &= Lu + \alpha^+\theta_{D^+}L(\theta_{D^-}u^+) + \alpha^-\theta_{D^-}L(\theta_{D^+}u^-) \\ &= Lu + \alpha^+L(\theta_{D^-}u^+) + \alpha^-L(\theta_{D^+}u^-) - \underbrace{\alpha^+\theta_{D^+}\Gamma L(\theta_{D^-}u^+)}_{=0} - \underbrace{\alpha^-\theta_{D^-}\Gamma L(\theta_{D^+}u^-)}_{=0} \\ &= Lu + L(\alpha^+\theta_{D^-}u^+ + \alpha^-\theta_{D^+}u^-). \end{aligned}$$

Hence, if $g = g_\alpha$ on the right-hand side of (2), then $v = u + \alpha^+\theta_{D^-}u^+ + \alpha^-\theta_{D^+}u^-$ and formula (10) holds. Uniqueness promptly follows from the solvability/uniqueness of the original problem. \square

Let us consider some implications of Theorem 2. If we set $\alpha^+ = \alpha^- = 0$ in (9), then we introduce no control, and $v = u$. The choice $\alpha^+ = 1, \alpha^- = -1$ is equivalent to (5) and corresponds to the complete shielding of D^+ . If $\alpha^+ = 0$ and $\alpha^- = -1$, then we again obtain a complete shielding of D^+ , however, in contrast to the previous example, the field v on D^- remains equal to u and thus unaffected by the controls. If $\alpha^+ = \alpha^- = -1$, then domains D^+ and D^- appear completely shielded from one another; this solution was also obtained in [17]. Clearly, by choosing other values of α^+ and α^- (in particular, fractional), we can achieve any desired degree of attenuation (or amplification) of the exterior field on a given subdomain.

If $\alpha^+ = -1$ and $\alpha^- = 1$, then we shield the domain $D^- \setminus \Gamma$ from the field generated in D^+ . This case corresponds to the control source

$$g_- = -\theta_{D^+}L(\theta_{D^-}u) = -g_+,$$

where g_+ is given by (5). Indeed,

$$g_+ + g_- = -L(\theta_{D^+}u) + \theta_{D^+}L(\theta_{D^+}u) - L(\theta_{D^-}u) + \theta_{D^-}\Gamma L(\theta_{D^-}u) = -f + f = 0.$$

As has been mentioned, the source term g_+ depends only on the Cauchy data (7). The same is obviously true for g_- . In other words, these control sources depend only on the total sound field and its derivatives at the interface Γ . However, for the general control g_α of (9) this is no longer true. The sources g^+ and g^- of (8) are determined by the fields u^+ and u^- , respectively. Hence, in addition to the total field we need to know the contribution from the sources in one of the

subdomains, either D^- or D^+ . This additional information can be extracted by first applying the controls g_+ or g_- and then conducting the second set of measurements at the interface, see [17].

3. Example: The Helmholtz equation

Time-harmonic acoustic field (pressure) in an isotropic homogeneous medium is governed by the Helmholtz equation with a constant wavenumber μ :

$$\Delta u + \mu^2 u = f,$$

which is a particular form of equation $Lu = f$ from (1). Then, the control g_+ is given by the linear combination of a single layer and a double layer source terms at the interface Γ , see [13,18]:

$$g_+ = \frac{\partial u}{\partial \mathbf{n}} \Big|_{\Gamma} \delta(\Gamma) + \frac{\partial u_{\Gamma} \delta(\Gamma)}{\partial \mathbf{n}}.$$

This control is fully determined by u_{Γ} and $\frac{\partial u}{\partial \mathbf{n}} \Big|_{\Gamma}$. To apply the general control g_{α} of (9), we, in addition, need to know u_{Γ}^+ and $\frac{\partial u^+}{\partial \mathbf{n}} \Big|_{\Gamma}$.

4. Discrete formulation of the AC problem

Consider a finite-difference counterpart of problem (1):

$$\begin{aligned} \sum_{n \in N_m} a_{mn} u_n &= f_m, \quad m \in M, \\ u_N &\in \mathcal{E}_N. \end{aligned} \tag{11}$$

Here, M is the grid for the right-hand side f_m ; N_m is the stencil associated with every node $m \in M$; a_{mn} , $m \in M$, $n \in N_m$, are the coefficients of the scheme; $N = \cup N_m$, $m \in M$, is the grid for the solution u_n ; \mathcal{E}_N is the space of grid functions $u_N = \{u_n\}$, $n \in N$, such that the solution of BVP (11) exists and is unique for any right-hand side $f_M = \{f_m\}$, $m \in M$. Inclusion $u_N \in \mathcal{E}_N$ of (11) therefore approximates the inclusion $u(\mathbf{x}) \in \mathcal{E}_D$ of (1).

Let us specify a subset $M^+ \subset M$ and define the sets: $M^- = M \setminus M^+$; $N^+ = \cup N_m$, $m \in M^+$; and $N^- = \cup N_m$, $m \in M^-$. The grid boundary γ is defined as $\gamma = N^+ \cap N^-$.

A grid function g_m , $m \in M$, is said to be a *discrete active control* if the solution v_n of the BVP

$$\begin{aligned} \sum_{n \in N_m} a_{mn} v_n &= f_m + g_m, \quad m \in M, \\ v_n &\in \mathcal{E}_N, \end{aligned} \tag{12}$$

satisfies some predetermined constraints either on N^+ or on N^- or on the entire N .

Similarly to what has been done in Section 2, we introduce

$$f_m^+ = \begin{cases} f_m, & \text{if } m \in M^+, \\ 0, & \text{if } m \in M^-, \end{cases} \quad \text{and} \quad f_m^- = \begin{cases} 0, & \text{if } m \in M^+, \\ f_m, & \text{if } m \in M^-, \end{cases}$$

and consider two BVPs formulated for the sound generated inside M^+ and M^- , respectively:

$$\begin{aligned} \sum_{n \in N_m^+} a_{mn} u_n^+ &= f_m^+, \quad m \in M^+, \\ u_N^+ &\in \mathcal{E}_N, \end{aligned} \tag{13}$$

and

$$\begin{aligned} \sum_{n \in N_m^-} a_{mn} u_n^- &= f_m^-, \quad m \in M^-, \\ u_N^- &\in \mathcal{E}_N. \end{aligned} \tag{14}$$

For the solutions of BVPs (11), (13), and (14), we obviously have $u_N = u_N^+ + u_N^-$.

Let us consider the following control function [cf. formula (5)]:

$$g_m = \begin{cases} 0, & \text{if } m \in M^+, \\ -\sum a_{mn} \bar{u}_n, & \text{if } m \in M^-, \end{cases} \quad \text{where } \bar{u}_n = \begin{cases} u_n, & \text{if } n \in \gamma, \\ 0, & \text{elsewhere.} \end{cases} \tag{15}$$

Then, the solution $v_N = \{v_n\}$, $n \in N$, of BVP (12) coincides on $N^+ \subset N$ with the solution of BVP (13), see [12]. Hence, the grid domain N^+ becomes completely shielded from the exterior sound.

In addition, introduce the control functions [cf. formulae (8)]:

$$g_m^+(\bar{u}_N^+) = \begin{cases} \sum a_{mn} \bar{u}_n^+, & \text{if } m \in M^+, \\ 0, & \text{if } m \in M^-, \end{cases} \quad \text{where } \bar{u}_n^+ = \begin{cases} u_n^+, & \text{if } n \in \gamma, \\ 0, & \text{elsewhere,} \end{cases} \tag{16}$$

and

$$g_m^-(\bar{u}_N^-) = \begin{cases} 0, & \text{if } m \in M^+, \\ \sum a_{mn} \bar{u}_n^-, & \text{if } m \in M^-, \end{cases} \quad \text{where } \bar{u}_n^- = \begin{cases} u_n^-, & \text{if } n \in \gamma, \\ 0, & \text{elsewhere,} \end{cases} \quad (17)$$

as well as their linear combination [cf. formula (9)]:

$$g_m(\alpha^+, \alpha^-) = \alpha^+ g_m^+(\bar{u}_N^+) + \alpha^- g_m^-(\bar{u}_N^-). \quad (18)$$

Note that to define the control (15) it is sufficient to know only the total field u_n at the grid boundary γ , whereas to define the controls (16), (17), and (18) we additionally need to know either u_n^+ or u_n^- on γ . This extra information can be retrieved by first applying the control (15) and then conducting the second set of measurements on γ , which will produce $v_n|_{n \in \gamma} = u_n^+|_{n \in \gamma}$.

Theorem 3. *The field due to both the primary source f_m and the control (18) is [cf. formula (10)]*

$$v_n = \begin{cases} u_n^+ + (1 + \alpha^-)u_n^-, & \text{if } n \in N^+ \setminus \gamma, \\ u_n^- + (1 + \alpha^+)u_n^+, & \text{if } n \in N^- \setminus \gamma, \\ (1 + \alpha^+)u_n^+ + (1 + \alpha^-)u_n^-, & \text{if } n \in \gamma. \end{cases} \quad (19)$$

Proof. Consider the function:

$$\hat{v}_n = \begin{cases} \alpha^- u_n^-, & \text{if } n \in N^+ \setminus \gamma, \\ \alpha^+ u_n^+, & \text{if } n \in N^- \setminus \gamma, \\ \alpha^+ u_n^+ + \alpha^- u_n^-, & \text{if } n \in \gamma, \end{cases} \quad (20)$$

so that $\hat{v}_n + u_n = v_n$, where u_n solves BVP (11) and v_n is given by (19). For \hat{v}_n of (20), we have:

$$\sum_{n \in N_m} a_{mn} \hat{v}_n = \begin{cases} \alpha^- \sum a_{mn} u_n^- + \alpha^+ g_m^+(u_N^+), & \text{if } m \in M^+, \\ \alpha^+ \sum a_{mn} u_n^+ + \alpha^- g_m^-(u_N^-), & \text{if } m \in M^-. \end{cases}$$

As BVP (12) is uniquely solvable, its solution for g_m given by (18) coincides with (19). \square

Theorem 3 is a discrete counterpart of Theorem 2, and it brings along a very similar set of implications. By choosing different values of α^+ and α^- , we can have N^+ or N^- completely shielded from their respective complements, or we can have both $N^+ \setminus \gamma$ and $N^- \setminus \gamma$ completely shielded from one another, or in general, we can have the exterior sound on each subdomain attenuated or amplified by a prescribed factor. It is important to note that according to (19), the fields on the domains $N^+ \setminus \gamma$ and $N^- \setminus \gamma$ do not depend on the values of α^+ and α^- , respectively. Hence, the control (16) affects only the field on N^+ , whereas the control (17) may alter the field only on N^- . Further details on the discrete setting can be found in [19].

5. Conclusions

We consider a steady-state or time-harmonic field governed by a linear PDE or system on a region composed of two subdomains separated by a common interface. The original sources of the field are located on both subdomains, and on the interface we introduce additional sources called controls. The controls are designed so that on each subdomain they enable a desired degree of cancellation (or intensification) of the field due to the sources on the complementary subdomain. The general solution for controls is obtained for both continuous and discrete formulation.

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