# Numerical Simulation of Focusing Nonlinear Waves in the Non-Paraxial Regime

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**Abstract**

We are solving the multi-D nonlinear Helmholtz equation that governs the propagation of intense laser light in Kerr dielectrics. Our numerical method addresses a key challenge in numerical optics, namely, singular behavior of solutions when the backscattered component of the electric field is neglected. It also guarantees high-order accuracy in the presence of material discontinuities. The key components of the method include a semi-compact finite-difference scheme, a nonlocal two-way artificial boundary condition, and a nonlinear Newton's solver. Our simulation provide the first ever direct numerical evidence that the singularity of the solution which characterizes the paraxial propagation of high power beams disappears once the non-paraxial effects are included.

## 1 Mathematical Model

The propagation of linearly polarized, continuous wave beams in isotropic Kerr media is governed by the scalar nonlinear Helmholtz equation (NLH):

$$E_{zz} + \Delta_\perp E + k_0^2 \left(1 + (2n_2/n_0)|E|^2\right) E = 0, \qquad (1)$$

where $E = E(z, \boldsymbol{x})$ is the electric field, $k_0$ is the linear wavenumber, $n_0$ is the linear index of refraction and $n_2$ is the Kerr coefficient. The impinging beam in (1) propagates along the coordinate $z$, and $\Delta_\perp$ denotes the Laplacian w.r.t. the remaining coordinate(s) $\boldsymbol{x}$.

The NLH (1) is a reduced model obtained from Maxwell's equations by a series of approximations. Yet its significance lies in the fact that it is the simplest model in nonlinear optics that enables the propagation of waves in all directions. In particular, it accounts for the important phenomenon of backscattering, which is due to the nonlinearly induced nonuniformities of the overall refraction index in (1).

Introducing the ansatz $E = e^{ik_0z}\phi$, where the envelope $\phi = \phi(z, \boldsymbol{x})$ is assumed to vary slowly compared with the fast carrier oscillation $e^{ik_0z}$, one can neglect the small $\phi_{zz}$ term (i.e., make the paraxial, or forward scattering, approximation), and reduce the NLH (1) to the nonlinear Schrödinger equation (NLS):

$$2ik_0\phi_z + \Delta_\perp\phi + 2k_0^2 n_2/n_0|\phi|^2\phi = 0. \qquad (2)$$

The NLS (2) is a canonical model in nonlinear optics for the propagation of intense laser beams in Kerr media. In particular, it captures the central phenomenon of nonlinear self-focusing. Yet, solutions to equation (2) can become singular (collapse) at a finite propagation distances if the input power exceeds a certain critical limit. The effect of collapse, however, is clearly non-physical because it assumes that a finite amount of power focuses to a point; besides, the paraxial approximation itself breaks down near the singularity. Therefore, a question of key importance is whether the singularity formation is arrested by "taking a step back" and employing the scalar NLH.

## 2 Numerical Method

We solve the NLH (1) for the plane-parallel slab of Kerr material immersed into a homogeneous linear medium. The Kerr slab itself can be layered, or grated, in which case $k_0$, $n_0$, and $n_2$ can all be piecewise constant functions of $z$. In doing so, continuity of the field $E$ and of its first derivative $E_z$ is assumed at all material interfaces. In the transverse direction(s), the properties of the medium remain smooth. The solution is driven by a single or multiple laser beam(s) that impinge on the Kerr slab either normally or at an angle.

Any discretization of the NLH (1) must be of high-order accuracy so as to minimize the number of points per wavelength required for solving equation (1) with sub-wavelength resolution on a large domain, and also help resolve the small-scale phenomenon of backscattering against the background of a forward-propagating wave. Besides, it must maintain accuracy across the material discontinuities.

The desirable properties are attained by a fourth-order semi-compact scheme built on a uniform rectangular grid aligned with the slab and the material interfaces. The geometry may be either Cartesian or cylindrical with $z$ being the axis. As there are no material discontinuities in the direction(s) orthogonal to $z$, the scheme uses standard central differences on a five-node stencil. In the direction $z$, the scheme employs a compact three-node stencil. It appears particularly convenient for handling the interfaces (enforcing the continuity conditions), and it also helps reduce the bandwidth of the final matrix.

The interior discretization is supplemented by nonlocal two-way artificial boundary conditions (ABCs) in the direction $z$, and by local radiation boundary conditions at the transverse far-field boundaries. The ABCs make the outer surfaces of the slab transparent for the outgoing radiation and at the same time correctly prescribe the given incoming beam(s). They are obtained by separating the variables in the linear region (right outside the Kerr slab) and subsequently selecting only the proper outgoing modes.

The resulting discrete system of nonlinear equations on the grid is solved by Newton's method. There is one nontrivial step in building the Newton's linearization. Namely, the nonlinearity in equation (1) is non-differentiable in the sense of Frechét for complex-valued solutions. Hence, equation (1) has first to be recast as a system of two real equations, after which Newton's linearization becomes possible.

## 3 Results

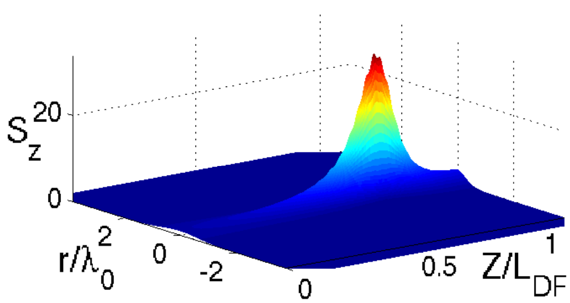


Figure 1: Arrest of collapse.

Figure 1 is an example showing that the formation of singularity is suppressed in the case of nonparaxial propagation governed by the NLH (1). The specific quantity plotted is the longitudinal energy flux, and the parameters of computation are chosen such that the corresponding NLS solution would have blown up, i.e., collapsed. Instead, in Figure 1 we see a rather sharp focusing of the beam followed by its defocusing. This is a clear indication that the waves' collapse is indeed arrested by nonparaxiality.

There is another group of interesting nonlinear wave phenomena that can be, and have been, simulated using the NLH (1). Namely, for the propagation in waveguides solutions to the NLS (2) exist in the form of spatial solitons that can sustain themselves and propagate over long distances essentially with no diffraction. However, it was not clear until our recent work whether similar solutions existed in the case of nonparaxial propagation.

Using the new numerical methodology, not only have we been able to compute the nonparaxial solitons, but also show that they can be very narrow in width, about one carrier wavelength $\lambda = 2\pi/k_0$. In contradistinction to that, the NLS (2) does not admit narrow solitons. We have also been able to compute interactions ("collisions") between counter-propagating spatial solitons, see Figure 2. We note that the NLH (1) appears a naturally well suited model for analyzing this configuration, whereas the NLS (2) entails some inherent mathematical difficulties related to the propagation in opposite directions.

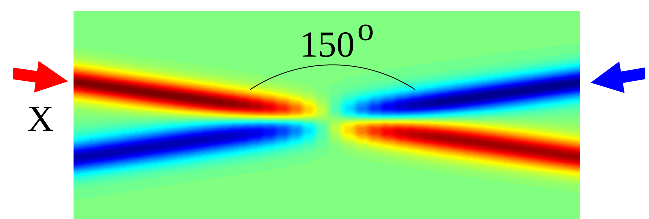


Figure 2: Collision of solitons.

## 4 Conclusions

For the first time in the literature, our work provides direct numerical evidence that nonparaxiality arrests the collapse of focusing nonlinear waves. Besides, and also for the first time ever, the new method helps compute the (narrow) nonparaxial solitons and their collisions. The computations are done with high-order accuracy, which is maintained in the presence of material discontinuities as well. More detail can be found in references [1], [2].

## References


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[2] G. Baruch, G. Fibich, and S. Tsynkov. High-order numerical solution of the nonlinear Helmholtz equation in multidimensional layered media. *J. Comput. Physics*, 228 (2009):3789–3815.