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Inverse source problem and active shielding for composite domains

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Abstract

The problem of active shielding (AS) for a multiply connected domain consists of constructing additional sources of the field (e.g., acoustic) so that all individual subdomains can either communicate freely with one another or otherwise be shielded from their peers. This problem can be interpreted as a special inverse source problem for the differential equation (or system) that governs the field. In the paper, we obtain general solution for a discretized composite AS problem and show that it reduces to solving a collection of auxiliary problems for simply connected domains.

Keywords: Multiply connected domain; Inverse source; Active shielding; Control of sound; Calderon's projection; Difference potentials method

1. Introduction

The classical inverse source problem is a problem of enabling a desired alteration in the solution of a given differential equation by means of appropriately modifying its source terms, or equivalently, by adding new sources. This problem has been studied extensively over the past three decades, both from the standpoint of physics/engineering, see, e.g., [1,2], as well as from the standpoint of mathematics, see, e.g., [3]. A particular type of alteration that may be desired in the solution is shielding of a given subdomain from the effect of the sources on the complementary domain. This problem is important for many applications. For example, in acoustics one is often interested in protecting a given region of space from the unwanted sound (i.e., noise) that originates from the sources outside of this region. As the protection, or shielding, is rendered by the specially constructed additional sources of sound (rather than, say, by insulation), it is called active shielding (AS).

In the acoustics literature, the AS problem is often referred to as the problem of active control of sound. The first theoretical publications on the subject belong to Jessel [4], Malyuzhinets [5], and Fedoryuk [6]. Several monographs and collection volumes have appeared over the years that provide a comprehensive review of the discipline [7–9]. Most theoretical approaches developed to date presume a fairly detailed knowledge of the sources of noise and of the

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properties of the sound-conducting medium, e.g., [10,11]. There are many methods that provide for a reduction of noise either at a collection of discrete locations, e.g., [12–15], or along some predetermined directions, e.g., [16,17].

In our previous work, we obtained a general solution of the AS problem in both continuous [18] and finite-difference formulation [19], see also [20,21]. Our approach employs the Calderon boundary projection operators and the difference potentials method [22]. In contradistinction to all other methods available in the literature, it requires minimum a priori information — only the knowledge of the overall solution (total acoustic field) on the boundary of the protected region. No information on either the actual distribution of the sources or the properties of the medium is needed. Moreover, one does not need to know the Green's function of the governing equation either, unlike, e.g., in [5], where the Green's function of the Helmholtz equation is required. In [23], we generalized our technique and obtained the continuous and discrete solution of the AS problem in the form of surface potentials. Further extensions, including optimization problems, are introduced and studied in [24–26].

The foregoing AS solution [18,19,23–26] was constructed for a simply connected region. In this paper, we extend it to the case of a composite (i.e., multiply connected) protected region. Moreover, we introduce *a key new element into the formulation.* Namely, the overall domain of the solution is arbitrarily split into a collection of subdomains, and the latter are selectively allowed to either communicate freely with one another or otherwise be shielded from their peers. In doing so, no reciprocity is assumed, i.e., for a given pair of subdomains one may be allowed to hear the other, but not vice versa.

In the core of the current paper, *a fundamental theorem is proven that provides a general solution of the composite AS problem* with a predetermined communication pattern between subdomains. It turns out that this solution can only be obtained in two stages. The preprocessing stage requires solving special auxiliary problems, which translates into additional computations and/or measurements in the practical context. Altogether, the solution of the composite AS problem is reduced to the solution of a series of subproblems that can all be addressed by our original methodology [18, 19].

2. Continuous formulation of the problem

Let the field u be a solution to the following linear boundary value problem:

$$Lu = f, \quad x \in D, \tag{1}$$

$$lu|_{\partial D} = 0. \tag{2}$$

In particular, u may be acoustic pressure, in which case L is the Helmholtz operator; u may also be a vector field with the acoustic pressure and velocity as components. For simplicity, the quantity u will hereafter be called sound.

Assume that the domain D consists of nonintersecting subdomains D_i, $\bar{D} = \cup \bar{D}_i$, $i = 1, 2, \ldots, I$. Introduce an $I \times I$ matrix α with the entries equal to either 0 or 1. If $\alpha_{ij} = 1$, then the field due to the sources on D_j is allowed on D_i. Otherwise, the field originating in D_j is considered adverse on D_i. Naturally, $\alpha_{ii} = 1$. At the same time, no reciprocity is assumed so that the matrix α is not necessarily symmetric. This problem will be referred to as the α–AS problem. It admits the following mathematical formulation.

Introduce a new boundary value problem similar to (1) and (2):

$$Lv = f + g, \quad x \in D, \tag{3}$$

$$lv|_{\partial D} = 0, \tag{4}$$

and a set of problems for $u^{(i)}(x)$ — the total allowable field on D_i:

$$Lu^{(i)} = f^{(i)}, \quad x \in D, \tag{5}$$

$$lu^{(i)}|_{\partial D} = 0, \tag{6}$$

where $f^{(i)}(x) = \alpha_{ij} f(x)$ if $x \in D_j$, $j = 1, 2, \ldots, I$. The function g is called an active α-control if on every D_i, $i = 1, 2, \ldots, I$, the solution of problem (3) and (4) coincides with the solution of the corresponding problem (5) and (6):

$$v(x) = u^{(i)}(x) \quad \text{if } x \in D_i,\ i = 1, 2, \ldots, I. \tag{7}$$

Thus, the α–AS problem is reduced to constructing such additional sources $g(x)$ that would eliminate the unwanted (i.e., adverse) sound for each domain D_i. Next, we consider a finite-difference formulation of the α–AS problem.

3. Discrete formulation of the problem

First, we introduce a finite-difference counterpart of problem (1) and (2):

$$\sum_{n\in N_m} a_{mn}u_n = f_m, \quad m \in M, \tag{8}$$

$$u_n \in U_N. \tag{9}$$

Here, M is the grid for the right-hand side f_m; N_m is the stencil associated with every node $m \in M$; a_{mn}, $m \in M$, $n \in N_m$, are the coefficients of the scheme; $N = \cup N_m$, $m \in M$, is the grid domain of the solution; U_N is a linear space of grid functions u_n, $n \in N$, such that the solution of problem (8) and (9) exists and is unique. Inclusion (9) approximates boundary condition (2).

Let the set M consist of $I \geq 2$ nonintersecting subsets M_i^+, $M = \cup M_i^+$, $i = 1, 2, \ldots, I$. Also, for every $i = 1, 2, \ldots, I$ let: $M_i^- = M\backslash M_i^+$, $N_i^+ = \cup N_m$ $(m \in M_i^+)$, $N_i^- = \cup N_m$ $(m \in M_i^-)$, and $\gamma_i = N_i^+ \cap N_i^-$. The set γ_i is called the grid boundary between N_i^+ and N_i^-, and the set $\gamma = \cup\gamma_i$, $i = 1, 2, \ldots, I$, is the overall grid boundary. Hereafter, we will assume that the solution u_n is known on γ. For example, the acoustic pressure can be measured by microphones.

It is clear that each point $n \in N\backslash\gamma$ may only belong to one grid subdomain $N_i^+ \subset N$. Otherwise, if $n \in \gamma$, it belongs to more than one subdomain. We will always assign each point $n \in N$ to one and only one subdomain N_i^+. When $n \in \gamma$, the subdomain can be selected either arbitrarily or based on some additional information that characterizes a given application.

Let $\bar{N}_i^+$ be the set of nodes assigned to N_i^+, $i = 1, 2, \ldots, I$. Consider problems

$$\sum_{n\in N_m} a_{mn}z_n^{(i)} = \alpha_{ij} f_m \quad \text{if } m \in M_j^+,\ i = 1, 2, \ldots, I, \tag{10}$$

$$z_n^{(i)} \in U_N. \tag{11}$$

Each problem (10) and (11) differs from (8) and (9) only by the right-hand side. For $n \in N$, define the grid function

$$z_n = z_n^{(i)} \quad \text{if } n \in \bar{N}_i^+,\ i = 1, 2, \ldots, I. \tag{12}$$

Similarly to the continuous case, we introduce the discrete active α-controls.

Definition 1. A grid function g_m, $m \in M$, is said to be an active α-control if the solution v_n of the problem

$$\sum_{n\in N_m} a_{mn}v_n = f_m + g_m, \quad m \in M, \tag{13}$$

$$v_n \in U_N, \tag{14}$$

coincides with the function z_n of (12):

$$v_n = z_n. \tag{15}$$

To build an active α-control g_m, we will need to know z_n of (12) for $n \in \gamma$; other values of z_n will not matter. In other words, we will need an additional procedure for solving problems (10) and (11), $i = 1, 2, \ldots, I$.

For every $i = 1, 2, \ldots, I$, we will now introduce two new subsets of grid nodes:

$$M^+[i] \stackrel{\text{def}}{=} \bigcup_{\alpha_{ij}=1} M_j^+, \qquad M^-[i] = M \setminus M^+[i],$$

i.e., $M^+[i]$ is a sum of all those sets M_j^+, $j = 1, 2, \ldots, I$, which contain the sources of the field admissible on the i-th subdomain ($\alpha_{ij} = 1$), and $M^-[i]$ is its complement. In addition, we define the following sets:

$$N^+[i] = \bigcup_{m\in M^+[i]} N_m, \qquad N^-[i] = \bigcup_{m\in M^-[i]} N_m, \qquad \gamma[i] = N^+[i]\bigcap N^-[i].$$

Then, problem (10) and (11) can be equivalently re-formulated as:

$$\sum_{n\in N_m} a_{mn} z_n[i] = \begin{cases} f_m, & \text{if } m \in M^+[i], \\ 0, & \text{if } m \in M^-[i], \end{cases} \tag{16}$$

$$z_n[i] \in U_N. \tag{17}$$

It was proven in work [19] that the solution of problem (16) and (17) at the nodes $n \in N^+[i]$ for a given i coincides with the solution of the following problem:

$$\sum_{n\in N_m} a_{mn} v_n[i] = f_m + g_m[i], \quad m \in M, \tag{18}$$

$$v_n[i] \in U_N, \tag{19}$$

where $g_m[i]$ are the auxiliary control sources defined as

$$g_m[i] = \begin{cases} 0, & \text{if } m \in M^+[i], \\ -\sum\limits_{n\in N_m} a_{mn} w_n, & \text{if } m \in M^-[i], \end{cases} \tag{20}$$

and w_n is a special auxiliary function:

$$w_n = \begin{cases} u_n, & \text{if } n \in \gamma[i], \\ 0, & \text{if } n \notin \gamma[i]. \end{cases} \tag{21}$$

Thus, by adding the sources $g_m[i]$ to the right-hand side f_m of the original problem (8) and (9), we can compute the values of $v_n[i] = z_n[i]$ at the nodes $n \in N^+[i]$, and in particular, at $n \in \gamma_i$, because $\gamma_i \subset N_i^+ \subset N^+[i]$. Indeed, problem (8) and (9) is assumed uniquely solvable and problem (18) and (19) only differs from it by the right-hand side. Once this process is repeated for all i, we obtain $z_n[i]$, $n \in \gamma_i$, $i = 1, 2, \dots, I$. Then the following theorem holds.

Theorem 1. *Consider a specific $z_n \in U_N$ defined by formula (12). There is a unique active α-control g_m, $m \in M$, in the sense of Definition 1:*

$$g_m = \sum_{n\in\gamma\cap N_m} a_{mn}(z_n|_\gamma - z_n[i]) \quad \text{for } m \in M_i^+, \quad \text{where} \quad z_n|_\gamma = z_n[j] \quad \text{if } n \in \gamma \cap \bar{N}_j^+. \tag{22}$$

Proof. Recall that according to formula (12) the function z_n is composed of the fragments of individual functions $z_n^{(j)}$ [formulae (10) and (11)] defined on the grid subsets $\bar{N}_j^+$. Let $m \in M_i^+$ for a particular i ($i = 1, 2, \dots, I$). Then, substituting z_n into the left-hand side of (8), we obtain:

$$\begin{aligned}\sum_{j=1}^{I} \sum_{n\in N_m\cap\bar{N}_j^+} a_{mn} z_n^{(j)} &= \sum_{j=1}^{I} \sum_{n\in N_m\cap\bar{N}_j^+} a_{mn} \left[z_n^{(i)} + \left(z_n^{(j)} - z_n^{(i)}\right)\right] \\ &= \sum_{n\in N_m} a_{mn} z_n^{(i)} + \sum_{j=1}^{I} \sum_{n\in N_m\cap\bar{N}_j^+} a_{mn}\left(z_n^{(j)} - z_n^{(i)}\right) \\ &= f_m + \sum_{j=1}^{I} \sum_{n\in N_m\cap\bar{N}_j^+} a_{mn}\left(z_n^{(j)} - z_n^{(i)}\right) = f_m + g_m.\end{aligned}$$

The last equality holds because on the one hand, there is obviously no contribution into the sum from the term $j = i$, and on the other hand, if $m \in M_i^+$ and $j \neq i$, then we may only have $z_n^{(j)} - z_n^{(i)} \neq 0$ for those $n \in N_m \cap \gamma$ that also satisfy $n \in N_m \cap \bar{N}_j^+$, i.e., g_m is indeed given by (22). Uniqueness of the α-control g_m (22) is a direct implication of the unique solvability of problem (8) and (9). □

By construction, to build an active α-control one only needs to know u_γ and the solutions of problems (18) and (19) on $\gamma[i]$. Besides, by analyzing formulae (20)–(22) one can see that they do not change when the given sources

f_m, $m \in M$, the space U_N, or the coefficients a_{mn} (for $n \notin N_m \cap \gamma$) undergo changes. All those changes may only affect the input for formulae (20)–(22), namely, the values of $u_n|_\gamma$ measured at the boundary and the values of $z_n|_\gamma$ calculated on γ. Recall that the values of $z_n|_\gamma$ also depend only on $u_n|_\gamma$. Finally, it is important to note that to obtain an active α-control one does not need to know the whole solution of every problem (10) and (11); only the fragments on the corresponding subsets $\bar{N}_j^+$ are required.

4. Conclusions

The problem of active shielding has been formulated for composite domains. This problem can be interpreted as an inverse source problem of a particular type. Its general solution was obtained in the finite-difference formulation. This solution allows all individual subdomains to either communicate freely with one another or otherwise be shielded from their peers. In doing so, no reciprocity is assumed, i.e., for a given pair of subdomains one may be allowed to hear the other but not necessarily vice versa. Moreover, the general solution requires no information about the sources of the field. To construct this general solution, one only needs to measure the field itself at the boundaries of the subdomains, and to solve some additional AS problems in the standard single domain formulation.

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