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Optimization of power in the problems of active control of sound

J. Lončarić [a,1], S.V. Tsynkov [b,*]

[a] Los Alamos National Laboratory, MS-D432, P.O. Box 1663, Los Alamos, NM 87545, USA
[b] Department of Mathematics and Center for Research in Scientific Computation (CRSC), North Carolina State University, Box 8205, Raleigh, NC 27695, USA

Abstract

We analyze the problem of suppressing the unwanted component of a time-harmonic acoustic field (noise) on a predetermined region of interest. The suppression is rendered by active means, i.e., by introducing the additional acoustic sources called controls that generate the appropriate anti-sound. Previously, we have obtained general solutions for active controls in both continuous and discrete formulation of the problem. We have also obtained optimal solutions that minimize the L_1 or L_2 norm of the control sources; the physical interpretation of the former being the overall absolute acoustic source strength.

In the current paper, we minimize the power required for the operation of the active control system. It turns out that the corresponding analysis necessarily involves interaction between the sources of sound and the surrounding acoustic field, which was not the case for either L_1 or L_2. Even though it may first seem counterintuitive, one can build a control system (a particular combination of surface monopoles and dipoles) that would require no power input for operation and would even produce a net power gain while providing the exact noise cancellation. This usually comes at the expense of having the original sources of noise produce even more energy.

Keywords: Noise cancellation; Active control sources; Volumetric and surface controls; General solution; Monopoles and dipoles; Radiation of waves; Complex-valued quantities; Load on the sources by the field; Net power gain

1. Introduction

In the simplest possible formulation, the problem of active control of sound is posed as follows. Let $\Omega \subset \mathbb{R}^n$ be a given domain (bounded or unbounded), and Γ be its boundary: $\Gamma = \partial\Omega$, where the dimension of the space n is either 2 or 3. Both on Ω and on its complement $\Omega_1 = \mathbb{R}^n \backslash \Omega$ we consider a scalar time-harmonic field $p = p(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^n$, governed by the inhomogeneous Helmholtz equation:

$$\boldsymbol{L}p \equiv \Delta p + k^2 p = f \tag{1}$$

* Corresponding author. Tel.: +1-919-515-1877; fax: +1-919-515-3798.
E-mail addresses: josip@lanl.gov (J. Lončarić), tsynkov@math.ncsu.edu (S.V. Tsynkov).
URL: http://www.math.ncsu.edu/ stsynkov.
[1] The first author was a research fellow at ICASE when this paper was written.

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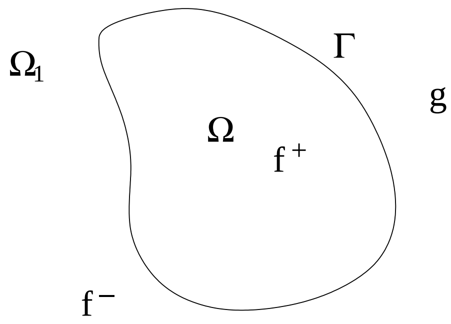

Fig. 1. Geometric setup.

Eq. (1) is subject to the Sommerfeld radiation boundary conditions:

$$p(\boldsymbol{x}) = \mathrm{O}(|\boldsymbol{x}|^{(1-n)/2}), \qquad \frac{\partial p(\boldsymbol{x})}{\partial |\boldsymbol{x}|} + \mathrm{i}kp(\boldsymbol{x}) = \mathrm{O}(|\boldsymbol{x}|^{(1-n)/2}), \quad \text{as } |\boldsymbol{x}| \to \infty \tag{2}$$

that specify the direction of wave propagation, and distinguish between the incoming and outgoing waves at infinity by prescribing the outgoing direction only. Boundary conditions (2) guarantee the unique solvability of the Helmholtz Eq. (1) for any compactly supported right-hand side $f = f(\boldsymbol{x})$. The physical meaning of the quantity $p(\boldsymbol{x})$ will be acoustic pressure, see Section 5, in which case the right-hand side to Eq. (1) will have the meaning of acoustic source density, see e.g. [1] for the definition. It is important to mention that as we are dealing hereafter with the traveling waves (radiation of sound toward infinity), all the corresponding solutions shall necessarily be complex-valued. Otherwise, it will be impossible to account for the key phenomenon of variation of phase with the change of spatial location.

The source terms $f = f(\boldsymbol{x})$ in Eq. (1) can be located on both Ω and its complement $\Omega_1 = \mathbb{R}^n \backslash \Omega$; to emphasize the distinction, we denote

$$f = f^+ + f^-, \qquad \text{supp } f^+ \subset \Omega, \qquad \text{supp } f^- \subset \Omega_1 \tag{3}$$

Accordingly, the overall acoustic field $p = p(\boldsymbol{x})$ can be represented as a sum of the two components:

$$p = p^+ + p^- \tag{4}$$

where p^+ is driven by the interior sources f^+, and p^- is driven by the exterior sources f^- with respect to Ω:

$$\boldsymbol{L}p^+ = f^+ \tag{5a}$$

$$\boldsymbol{L}p^- = f^- \tag{5b}$$

Note, both $p^+ = p^+(\boldsymbol{x})$ and $p^- = p^-(\boldsymbol{x})$ are defined on the entire $\mathbb{R}^n$, the superscripts "+" and "−" refer to the sources that drive each of the field components rather than to the domains of these components. The setup described above is schematically shown in Fig. 1 for the case of a bounded domain Ω. It is also important to mention that even though we do employ the notions of f^+ and f^- in order to provide a comprehensive problem setup, no explicit knowledge of either of these two source components will be needed for achieving our primary objective of noise control. As Section 2 of the paper indicates, we will only need to know the overall acoustic field and its normal derivative on the surface Γ.

Hereafter, we will call the component p^+ of (4), (5a) *sound*, or "friendly" part of the total acoustic field; the component p^- of (4) and (5b) will accordingly be called *noise*, or "adverse" part of the total acoustic field. In the formulation that we are presenting, Ω will be a predetermined region of space to be protected from noise. This means that we would like to eliminate the noise component of $p(\boldsymbol{x})$ inside Ω, while leaving the sound component there unaltered. In the mathematical framework that we have adopted, the component p^- of the total acoustic field, i.e., the response to the adverse sources f^- (see (3), (4), (5a) and (5b)), will have to be canceled out on Ω, whereas the component p^+, i.e., the response to the friendly sources f^+, will have to be left unaffected on Ω. A physically more involved but conceptually easy to understand example that can be given to illustrate the foregoing idea, is that inside the passenger compartment of an aircraft we would like to eliminate the noise coming from the propulsion system located outside the fuselage, while not interfering with the ability of the passengers to listen to the inflight entertainment programs or simply converse.

The concept of *active noise control* that we will be discussing implies that the component p^- is to be suppressed on Ω by introducing additional sources of sound $g = g(\boldsymbol{x})$ exterior with respect to Ω, $\operatorname{supp} g \subset \Omega_1$, so that the total acoustic field $\tilde{p} = \tilde{p}(\boldsymbol{x})$ be now governed by the equation (cf. Eq. (1)):

$$\boldsymbol{L}\tilde{p} = f^+ + f^- + g \tag{6}$$

and coincide with only the friendly component p^+ on the domain Ω:

$$\tilde{p}|_{\boldsymbol{x}\in\Omega} = p^+|_{\boldsymbol{x}\in\Omega} \tag{7}$$

The new sources $g = g(\boldsymbol{x})$ of (6), see Fig. 1, will hereafter be referred to as the *control sources* or simply *controls*. An obvious solution for these control sources is $g = -f^-$. This solution, however, is not always the most convenient one for either analysis or implementation, because on one hand, it requires an explicit and detailed knowledge of the structure and location of the sources f^-, which is, in fact, superfluous, see [2]. On the other hand, its implementation in many cases, like in the previously mentioned example with an airplane, may not be feasible. Fortunately, there are other solutions of the foregoing noise control problem (see Section 2, as well as our previous work [2] for detail), and some of them may be preferable from both the theoretical and practical standpoint. Besides, having a variety of solutions for the control sources allows us to formulate and solve optimization problems [3,4], as well as the power optimization problem analyzed hereafter.

To conclude the introduction, let us only mention that the area of active control of sound has a rich history of development, both as a chapter of theoretical acoustics, and in the perspective of many different applications. It is impossible to adequately overview this extensive area in the framework of a focused research publication. As such, we simply refer the reader to the monographs [5–7] that, among other things, contain a detailed survey of the literature. Potential applications for the active techniques of noise control range from the aircraft industry to manufacturing industry (reducing the machinery noise) to air and ground transportation (protecting residences from highway noise) to medicine (protecting patients from high levels of periodic noise generated by resonance coils in magnetic resonance imaging (MRI) machines) to the military to consumer products and other fields, including even such highly specialized and narrow areas as acoustic measurements in the wind tunnels. It is generally known that active techniques are more efficient for lower frequencies, and they are usually expected to complement passive strategies (sound insulation, barriers, etc.) that are more efficient for higher frequencies, because the rate of sound dissipation due to viscosity of the medium and heat transfer is proportional to the square of the frequency [8].

Let us also note that in the current paper we focus on the case of the standard constant-coefficient Helmholtz Eq. (1), which governs the acoustic field throughout the entire space $\mathbb{R}^n$. This allows us to make the forthcoming analysis most straightforward. However, one can consider other, more complex, cases as well, that involve variable coefficients, different types of far-field behavior, discontinuities in the material properties, and maybe even nonlinearities in the governing equations over some regions. Approaches to obtaining solutions for active controls in these cases are based on the theory of generalized Calderon's potentials and boundary projections, and can be found in our previous paper [2] and in the monograph by Ryaben'kii ([9], Part VIII).

2. General solutions for control sources

A general solution for the volumetric continuous control sources $g = g(\boldsymbol{x})$ is given by the following formula ($\Omega_1 = \mathbb{R}\backslash\Omega$):

$$g(\boldsymbol{x}) = -\boldsymbol{L}w|_{\boldsymbol{x}\in\Omega_1} \tag{8}$$

where $w = w(\boldsymbol{x})$, $\boldsymbol{x} \in \Omega_1$, is a special auxiliary function-parameter that parameterizes the family of controls (8). The function $w(\boldsymbol{x})$ must satisfy the Sommerfeld boundary conditions (2) at infinity, and at the interface Γ, the function w and its normal derivative have to coincide with the corresponding quantities that pertain to the total acoustic field p given by formula (4):

$$w|_\Gamma = p|_\Gamma, \qquad \left.\frac{\partial w}{\partial \boldsymbol{n}}\right|_\Gamma = \left.\frac{\partial p}{\partial \boldsymbol{n}}\right|_\Gamma \tag{9}$$

Other than that, the function $w(\boldsymbol{x})$ used in (8) is arbitrary, and consequently formula (8) defines a large family of control sources, which provides ample room for optimization. In particular, the obvious solution for controls $g = -f^-$ that we have mentioned previously fits into the framework of formulae (8), (9) by choosing $w(\boldsymbol{x}) = p(\boldsymbol{x})$, $\boldsymbol{x} \in \Omega_1$. Indeed, this $w(\boldsymbol{x})$ obviously satisfies boundary conditions (9), and $\boldsymbol{L}w = \boldsymbol{L}p = \boldsymbol{L}(p^+ + p^-) = f^+ + f^- = f^-$ because $f^+(\boldsymbol{x}) = 0$ for $\boldsymbol{x} \in \Omega_1$.

The justification for formula (8) as general solution for controls can be found in [2]. In our recent paper [3] we also emphasize that the controls

$$g(\boldsymbol{x}) = \int g(\boldsymbol{y})\delta(\boldsymbol{x} - \boldsymbol{y})\,\mathrm{d}\boldsymbol{y} = g * \delta$$

given by (8) are *actually volumetric control sources of the monopole type* with regular density $g \in L_1^{(\mathrm{loc})}(\mathbb{R}^n)$ (assuming that $w(\boldsymbol{x})$ was chosen sufficiently smooth so that to guarantee local absolute integrability of $g(\boldsymbol{x})$).

The control sources (8) possess several important properties. First of all, we see that to obtain these controls one needs no knowledge of the actual exterior sources of noise f^-. In other words, neither their location, nor structure, nor strength are required. According to (9), all one needs to know is p and $\partial p/\partial \boldsymbol{n}$ on the perimeter Γ of the protected region Ω. In a practical setting, $p|_\Gamma$ and $\partial p/\partial \boldsymbol{n}|_\Gamma$ can be interpreted as measurable quantities that are supplied to the control system as the input data. Moreover, as formulae (8) and (9) indicate and the analysis of [2] corroborates, these measurable quantities can refer to the overall acoustic field p, rather than only its unwanted component p^-. In other words, the methodology can automatically distinguish between the signals coming from the exterior and interior sources, and can tune the controls so that they cancel only the unwanted exterior signal. This capability, which essentially implies that the control sources (8) are insensitive to the interior sound $p^+(\boldsymbol{x})$, is extremely important because in many applications the overall acoustic field always contains a component that needs to be suppressed along with the part that needs to be left intact. Let us also note that a more general analysis of [2] based on Calderon's potentials and boundary projections yields the same formula for controls (8) and (9) for the cases that may involve variations in material properties and alternative types of the far-field behavior. Of course, the operator $\boldsymbol{L}$ will be a new variable-coefficient operator, and the function-parameter $w(\boldsymbol{x})$ will have to satisfy new far-field boundary conditions instead of the Sommerfeld boundary conditions.

Along with the volumetric controls (8), on can also consider *surface controls*, i.e., the control sources that are concentrated only on the interface Γ. A general solution for the surface controls is given by

$$g^{(\text{surf})} = -\left[\frac{\partial w}{\partial \boldsymbol{n}} - \frac{\partial p}{\partial \boldsymbol{n}}\right]\bigg|_{\Gamma} \delta(\Gamma) - \frac{\partial}{\partial \boldsymbol{n}}([w-p]|_{\Gamma}\delta(\Gamma)) \tag{10}$$

where $w = w(\boldsymbol{x})$, as before, denotes the auxiliary function-parameter. In contradistinction to the previous case, now it has to satisfy the homogeneous Helmholtz equation on the complementary domain: $\boldsymbol{L}w = 0$ for $\boldsymbol{x} \in \Omega_1$, and the Sommerfeld boundary conditions (2) at infinity, but at the interface Γ it may be arbitrary, i.e., it does not have to meet boundary conditions (9). The corresponding differences that are denoted by expressions in rectangular brackets in formula (10) and evaluated along Γ drive the surface control sources.[2] The first term on the right-hand side of (10) represents the density of a single-layer potential, which is a layer of monopoles on the interface Γ, and the second term on the right-hand side of (10) represents the density of a double-layer potential, which is a layer of dipoles on the interface Γ.

A detailed justification of formula (10) as general solution for surface controls can be found in [10], see also [3]. Here we only mention that the control output from $g^{(\text{surf})}$, i.e., the acoustic field that corresponds to the surface excitation (10) is given by

$$p^{(\text{surf})}(\boldsymbol{x}) = \begin{cases} -p^-(\boldsymbol{x}), & \boldsymbol{x} \in \Omega \\ -(w(\boldsymbol{x}) - p^+(\boldsymbol{x})), & \boldsymbol{x} \in \Omega_1 \end{cases} \tag{11}$$

Consequently, the surface controls (10) always cancel out the unwanted noise $p^-(\boldsymbol{x})$ on Ω. Their fundamental properties are the same as those of the volumetric controls (8). They are insensitive to the interior sound $p^+(\boldsymbol{x})$, see (11), and do not require any knowledge of the actual sources of noise f^-.

In the family of surface controls (10) we identify two important particular cases. First, the cancellation of $p^-(\boldsymbol{x})$, $\boldsymbol{x} \in \Omega$, can be achieved by using surface monopoles only, i.e., by employing only a single-layer potential as the annihilating signal (anti-sound). To do that, we need to find $w(\boldsymbol{x})$, $\boldsymbol{x} \in \Omega_1$, such that there will be no discontinuity on Γ between $p(\boldsymbol{x})$ and $w(\boldsymbol{x})$, i.e., in the function itself, and the discontinuity may only "reside" in the normal derivative (see formula (10)). This $w(\boldsymbol{x})$ will obviously be a solution of the following external Dirichlet problem:

$$\boldsymbol{L}w = 0, \qquad \boldsymbol{x} \in \Omega_1, \qquad w|_{\Gamma} = p|_{\Gamma} \tag{12}$$

subject to the Sommerfeld radiation boundary conditions (2). Problem (12) is always uniquely solvable on $\Omega_1 = \mathbb{R}^n \backslash \Omega$. Second, one can employ only the double-layer potential to cancel out $p^-(\boldsymbol{x})$, $\boldsymbol{x} \in \Omega$, i.e., use only surface dipoles as the control sources. In this case, the function $w(\boldsymbol{x})$, $\boldsymbol{x} \in \Omega_1$, has to be chosen

[2] These differences can actually be interpreted as surface discontinuities of the overall function defined on $\mathbb{R}^n$ and equal to $p(\boldsymbol{x})$ on Ω and to $w(\boldsymbol{x})$ on Ω_1.

such that the discontinuity on Γ be only in the function itself, i.e., between the actual values of $p(\boldsymbol{x})$ and $w(\boldsymbol{x})$, and not between the normal derivatives. This $w(\boldsymbol{x})$ should then solve the following external Neumann problem:

$$\boldsymbol{L}w = 0, \qquad \boldsymbol{x} \in \Omega_1, \qquad \left.\frac{\partial w}{\partial \boldsymbol{n}}\right|_{\Gamma} = \left.\frac{\partial p}{\partial \boldsymbol{n}}\right|_{\Gamma} \tag{13}$$

again, subject to the Sommerfeld conditions at infinity (2), which guarantee the solvability of (13).

We therefore see that surface control sources (10) are given by combinations of the monopole and dipole layers, with the two "extreme" cases corresponding to either only monopoles, see (12), or only dipoles, see (13). Other special cases of surface controls that are particularly interesting from the standpoint of acoustics will be identified in Section 4.

Altogether, we have now introduced active controls of two different types on the surface, but only one type of controls throughout the volume—monopoles, see formulae (8) and (9). This is not accidental. Let us note that from the standpoint of physics and engineering, the monopole and dipole sources provide different types of excitation to the surrounding sound-conducting medium. A point monopole source can be interpreted as a vanishingly small pulsating sphere that radiates acoustic waves symmetrically in all directions, whereas a dipole source resembles a small oscillating membrane that has a particular directivity of radiation. Moreover, in the genuine time-dependent acoustic context one can show that monopole sources are those that alter the balance of mass in the system, they are scalar in nature and reside on the right-hand side of the acoustic continuity equation, whereas dipole sources alter the balance of force, they are vectors and reside on the right-hand side of the acoustic momentum equation, see our recent work [3] for detail. This distinction basically warrants a separate consideration of the monopole and dipole type sources as far as the point-wise or surface excitation may be concerned. As, however, has been shown in [3], in the framework of time-harmonic volumetric excitation a separate consideration of dipole fields appears superfluous. In fact, any volumetric distribution of dipoles can, under the assumption of sufficient regularity, be recast in the form of an equivalent volumetric distribution of monopoles. In so doing, the dipole sources enter the right-hand side of the Helmholtz Eq. (1) through a divergence operator, whereas monopoles enter this right-hand side directly (up to a multiplicative constant). We refer the reader to our paper [3], as well as to the monograph [5], for further detail.

Let us also mention that from the standpoint of practice, a similar discrete formulation of the noise control problem may often be more convenient. This formulation has, in fact, been previously developed and analyzed in the context of finite differences, see papers [11,12] and monograph ([9], Part VIII) for detail; a brief description of the discrete formulation is also available in our papers [3,10]. In the current paper, however, we conduct the analysis only on the continuous level. As such, no discrete apparatus will be needed.

3. Strategies of optimization

Once the general solution for controls is available, either volumetric (8) or surface (10), the next step is to decide what particular element of this large family of functions will be optimal for a specific setting. There is a multitude of possible criteria for optimality that one can use. In many practical problems the cancellation of noise is only approximate and as such, the key criterion for optimization (or sometimes, the key constraint) is the quality of this cancellation, i.e., the extent of noise reduction. In contradistinction to that, in this paper we are considering ideal, or exact, cancellation, i.e., every particular control field from either volumetric (8) or surface (10) family completely eliminates the unwanted noise on the domain of interest. Consequently, the criteria for optimality of the controls that we can employ will not include the level of the residual noise as a part of the corresponding function of merit, and should rather depend only on the control sources themselves.

In our previous work, we have already considered two different criteria for optimization. In [3], we have optimized the L_1 norm of the volumetric control sources. In physical terms, this is equivalent to minimization of the overall absolute acoustic source strength, see [1,5], of the controls. The corresponding optimum happens to be the layer of surface monopoles given by formula (10) when the auxiliary function $w(\boldsymbol{x})$ is obtained by solving problem (12). In [4], we have optimized the L_2 norm of the control sources. The latter criterion does not have such a transparent physical meaning as the L_1 norm has, but the minimum is easier to compute numerically, including the cases that involve additional equality type constraints originating form geometric definitions.

In the current paper, we minimize the power required for the operation of the control system. It turns out that the corresponding analysis necessarily involves interaction between the sources of sound and the surrounding acoustic field, the phenomenon that is often referred to as *load* by the field on the sources, see [5]. We note that previously there was no need of considering the acoustic load in either L_1 or L_2 framework. We also note that previously we have optimized the volumetric monopoles, and in the L_1 case the optimum reduced to surface monopoles, whereas for L_2 it was still a particular volumetric distribution of sources. In contradistinction to that, in the current paper we optimize the power across both the monopoles and dipoles, but we are only considering the surface control sources (10) as the space for optimization. On one hand, having monopoles and dipoles together is imperative for finding the power optima. On the other hand, considering only surface controls still leaves the problem amenable to theoretical analysis.

Our key finding that we report in the current paper may first seem counterintuitive, but in fact, one can build a control system (i.e., a particular combination of surface monopoles and dipoles) that would require no power input for operation and would even produce a net power gain while providing for the exact noise cancellation. This is, in fact, not a paradox and not a perpetuum mobile. First of all, one can even imagine a passive system with similar properties that by definition does not need any power, e.g., a perfectly rigid barrier along the interface Γ covered with an ideal sound absorbing material from inside. This barrier will reflect (i.e., block) all the exterior noise and absorb all the interior sound thus leaving the component u^+ on Ω unaltered whilst completely eliminating u^-. In general, when an active setting is discussed, the conservation of energy does, of course, hold, but only when the entire closed system is analyzed, which must also include the original sources of noise. Then, what happens is that while cancelling the unwanted noise on Ω the controls alter the acoustic field on Ω_1 as well. This new field component produced by the controls on the domain Ω_1 impinges on noise sources f^- and makes them "work harder" and generate even more energy (this is called acoustic load [1]). Subsequently, a part of the additional energy is absorbed by the control system, which looks like the net power gain if the controls are considered alone, and the remaining part is radiated toward infinity.

4. Roadmap for power optimization

Exact volumetric noise cancellation can be achieved in a variety of ways. Perhaps the easiest to envision is the neutral control strategy, where the exterior acoustic field is unchanged while the interior field is due to interior sources only, i.e., without the unwanted noise. According to formula (11), this approach would correspond to setting $w(\boldsymbol{x}) \equiv p^+(\boldsymbol{x})$, $\boldsymbol{x} \in \Omega_1$, in the definition of surface controls (10).

A particularly interesting case is the control which fully decouples interior and exterior domains, such that on each subdomain the acoustic field is due solely to the sources in that domain. This control distribution corresponds to selecting $w(\boldsymbol{x}) = 2p^+(\boldsymbol{x})$, $\boldsymbol{x} \in \Omega_1$, in the definition (10). Then, the surface control output $p^{(\mathrm{surf})}(\boldsymbol{x})$ of (11) is equal to $-p^-(\boldsymbol{x})$ on Ω and $-p^+(\boldsymbol{x})$ on Ω_1. Under these conditions, the controls absorb all of the power emitted by the interior sources, while the power flux[3] due to the exterior sources is zero. Indeed, Γ is a closed surface, and the field right outside Γ in this case is $p^-(\boldsymbol{x})$, which is due to the exterior sources only, as there are no sources of the field $p^-(\boldsymbol{x})$ inside Γ. As such, there cannot be any power loss nor gain within Γ^-, so the power flux due to the exterior sources across this contour has to be zero. For the particular case of a circular perimeter Γ, this also follows from Lemma 1, see Section 7. This "decoupling control" is passive in the sense that it never requires net power to achieve its ends, and can actually deliver power absorbed from the interior sound field.

However, this is not all. While the interior sound field must only be p^+ due to our exact volumetric noise cancellation constraint, see formula (7), this still allows for control strategies which increase power extraction from the acoustic field. For example, if the controls alter the exterior sound field to increase acoustic loading on the noise sources, it becomes possible to extract increased amounts of power from them. In simplified terms, the exact noise cancellation is an affine constraint on the control monopoles and dipoles, while power is a quadratic form with a particular minimum within this affine constraint. Our approach shall be to explicitly calculate this minimum for simple geometries and to discuss the results.

5. Control power

Let us consider the case where a layer of monopole and dipole control sources is placed along the perimeter Γ. To compute the control power requirements, we need to surround Γ with two contour surfaces placed just inside and just outside of Γ and denoted by Γ^+ and Γ^-, respectively. We shall assume that these contour surfaces can be placed sufficiently close to Γ to ensure that they envelop only the controls and exclude all other sound and noise sources. The main physical quantities of interest in this section are the total pressure and velocity. Sound velocity fields can be written in terms of the velocity potential:

$$\boldsymbol{v} = \nabla\phi \tag{14}$$

where ϕ and p can be related using Euler's equation so that

$$p = -\rho\frac{\partial\phi}{\partial t} \tag{15}$$

[3] We use the term "power" to denote the total acoustic power leaving a particular domain, while the more specific term "power flux" denotes the portion of the acoustic power leaving the domain across a particular section of its boundary. Both terms represent power. This terminology follows the mathematical definition of the flux of a vector field, where the vector field itself measures power flux intensity and has different units (*power/area*).

The expression for the control power requirements (see [8], Chapter VIII) is the following time average, taken over the period $T = 2\pi/\omega$:

$$W_\omega = \frac{1}{T}\int_0^T \oint_{\Gamma^+\bigcup\Gamma^-} p(\boldsymbol{x},t)\boldsymbol{v}(\boldsymbol{x},t)\cdot \mathrm{d}\boldsymbol{a}\,\mathrm{d}t \tag{16}$$

where the total pressure and velocity fields consist of contributions of sound sources inside of Γ^+, control sources along Γ, and noise sources outside of Γ^-. While we had previously denoted this quantity by $\tilde{p}$, see formulae (6) and (7), using simplified notation which includes the contribution of controls in p will make the subsequent discussion easier to follow. The interior integral in expression (16) is the flux of the corresponding vector field through the surface $\Gamma^+\bigcup\Gamma^-$, and $\mathrm{d}\boldsymbol{a}$ denotes the oriented surface element: $\mathrm{d}\boldsymbol{a} = \boldsymbol{n}\,\mathrm{d}S$.

Since physical quantities are real quantities with frequency ω, their Fourier transform in time is conjugate symmetric, i.e., $p_\omega = \overline{p_{-\omega}}$. Upon expressing $\boldsymbol{v}$ in terms of p via formulae (14), (15), and the time-harmonic assumption, and then simplifying, expression (16) evaluates to

$$W_\omega = \frac{2}{\rho\omega}\oint_{\Gamma^+\bigcup\Gamma^-} \mathrm{Im}\left(p_\omega \frac{\partial \overline{p_\omega}}{\partial \boldsymbol{n}}\right)\cdot \mathrm{d}\boldsymbol{a} \tag{17}$$

where the complex field p represents the total pressure due to all sources and controls at frequency $\omega > 0$. We note that this expression accounts for both frequencies (ω and $-\omega$).

To obtain exact noise cancellation, we shall require that along Γ^+ (just inside Γ), pressures and velocities must coincide with the sound field produced by the interior sources only, see formula (7). Therefore, the portion of the contour integral along Γ^+ is fixed by the power emitted by the sound sources inside Γ^+ and cannot be changed by our choice of control sources. However, we still have the freedom to modify the portion of the contour integral (17) along Γ^-. We shall exploit this freedom to minimize the control power requirements (subject to additional constraints to be introduced later).

In order to explicitly evaluate the power requirements, let us consider a specific example which can be solved analytically. We shall focus on cancelling noise over a circular region. In this simple geometry the 2D problem can be decomposed into an infinite series of 1D problems.

6. Fourier decomposition

The decomposition is obtained by employing the Fourier transform in the circumferential direction. In so doing, we assume that the interface Γ is a circle of radius R. Due to orthogonality, the only terms that contribute to integral (17) in the Fourier representation come from products of Fourier coefficients whose index vectors add up to zero (e.g., (ω, m) and $(-\omega, -m)$). Since pressure is a real variable, its Fourier coefficients $p_{(\omega,m)}$ and $p_{(-\omega,-m)}$ are conjugates of each other. For $m = 0$, we have $\overline{p_{(\omega,0)}} = p_{(-\omega,0)}$; while for $m > 0$, this generates two conjugate symmetries: $\overline{p_{(\omega,m)}} = p_{(-\omega,-m)}$ and $\overline{p_{(\omega,-m)}} = p_{(-\omega,m)}$.

The total power can be written as an infinite sum of power terms indexed by m. Similarly, the exact noise cancellation constraint evaluates to an infinite collection of constraints for each m independently. The constrained optimization problem we intend to solve can be converted into an infinite sequence of simpler problems, indexed by m. The power (17) required for exact noise cancellation consists of two contributions. When exact noise cancellation is achieved, the portion of the integral along Γ^+ reduces to

$$W_\omega^+ = \frac{2}{\rho\omega}\int_{\Gamma^+} \mathrm{Im}\left(p^+ \frac{\partial\overline{p^+}}{\partial \boldsymbol{n}}\right)\cdot \mathrm{d}\boldsymbol{a} = -\frac{4\pi R}{\rho\omega}\sum_{m=-\infty}^{\infty} \mathrm{Im}\left(p^+_{(\omega,m)} \frac{\mathrm{d}\overline{p^+_{(\omega,m)}}}{\mathrm{d}r}\right)\Bigg|_{\Gamma^+}$$

which is fully determined by the interior sound sources and does not offer the possibility for control power optimization.

The second contribution comes from the portion of the contour integral along Γ^-, and this contour integral can be written in terms of the Fourier coefficients of the total pressure as follows:

$$W_\omega^- = \frac{4\pi R}{\rho\omega}\sum_{m=-\infty}^{\infty} \mathrm{Im}\left(p_{(\omega,m)} \frac{\mathrm{d}\overline{p_{(\omega,m)}}}{\mathrm{d}r}\right)\Bigg|_{\Gamma^-}$$

where we have used the conjugate symmetry properties of the Fourier coefficients of p. Let us also define the following form

$$\langle f(r), g(r)\rangle \stackrel{\text{def}}{=} \mathrm{Im}\left(f(r)\frac{\mathrm{d}\overline{g(r)}}{\mathrm{d}r}\right) \tag{18}$$

which will be useful in simplifying the discussion that follows. In summary, the control power is given by

$$W_\omega = W_\omega^- + W_\omega^+ = \frac{4\pi R}{\rho\omega}\sum_{m=-\infty}^{\infty}[\langle p_{(\omega,m)}(r), p_{(\omega,m)}(r)\rangle|_{\Gamma^-} - \langle p^+_{(\omega,m)}(r), p^+_{(\omega,m)}(r)\rangle|_{\Gamma^+}]$$

For reasons to be discussed later, *we shall seek to minimize control power only for* $|m| \le M$, *and require exact absorption of interior sound power* for $|m| > M$. Thanks to the fact that constraints decouple into an infinite sequence of equalities indexed by m, the control power can be optimized by choosing each $p_{(\omega,m)}$ individually. We shall solve this problem explicitly.

In the cylindrical coordinates, the Helmholtz Eq. (6) assumes the following form:

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + \frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2} + k^2 p = f \tag{19}$$

and upon Fourier transform

$$\frac{\mathrm{d}^2 p}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}p}{\mathrm{d}r}\left(k^2 - \frac{m^2}{r^2}\right)p = f$$

For simplicity, the circumferential Fourier mode index m and the frequency ω are usually implicit in our notation. Using this convention, the interpretation of variables should be based on the context in which they appear.

The Green's function for the problem (19) with Sommerfeld's boundary conditions (2), as obtained in [2] Eq. (5.5), has the following Fourier representation:

$$G(r,s) = \begin{cases} \frac{1}{4}J_m(kr)(Y_m(ks) + \mathrm{i}J_m(ks)) = -\frac{1}{4\mathrm{i}}J_m(kr)H_m^{(2)}(ks) & \text{for } r < s \\ \frac{1}{4}J_m(ks)(Y_m(kr) + \mathrm{i}J_m(kr)) = -\frac{1}{4\mathrm{i}}J_m(ks)H_m^{(2)}(kr) & \text{for } r \ge s \end{cases}$$

This Green's function allows us to explicitly write the decomposition of pressure along Γ^- into the part due to exterior noise and the part due to a combination of interior sources and controls, parameterized by a constant c, valid in a neighborhood of Γ^- which excludes all other sources:

$$p(r) = p^-(R)\frac{J_m(kr)}{J_m(kR)} + cH_m^{(2)}(kr) \quad \text{near } \Gamma^- \tag{20}$$

7. Power optimal exterior pressure

As we have already shown, W_ω^+ is fixed by the interior sound sources. In this section, we shall derive the power optimal form of W_ω^- for a particular m. We begin by stating the following lemma:

Lemma 1. *The following expressions using the form (18) hold along Γ^-:*

$$\langle \alpha J_m(kr), \alpha J_m(kr)\rangle = 0$$

$$\langle cH_m^{(2)}(kr), cH_m^{(2)}(kr)\rangle = \frac{2|c|^2}{\pi r}$$

$$\langle \alpha J_m(kr), cH_m^{(2)}(kr)\rangle + \langle c_m^{(2)}(kr), \alpha J_m(kr)\rangle = \frac{2Re(c\bar{\alpha})}{\pi r}$$

The above lemma is proven by explicit evaluation using properties of the Bessel functions.

The expression for power flux across Γ^- at particular ω and m (implicit in our simplified notation) can now be written explicitly

$$W^- = \frac{4\pi R}{\rho\omega}\langle p(r), p(r)\rangle|_{r=R} = \frac{8}{\rho\omega}\left(|c|^2 + Re\left(c\frac{\overline{p^-(R)}}{J_m(kR)}\right)\right) \tag{21}$$

This finally allows us to state and prove the following theorem:

Theorem 2. *Control power requirements for particular ω and m are minimized when the parameter c in (20) is*

$$c = -\frac{p^-(R)}{2J_m(kR)} \tag{22}$$

so that the corresponding power requirement contribution is

$$W^- = -\frac{2|p^-(R)|^2}{\rho\omega J_m(kR)^2}$$

and the total acoustic pressure just outside Γ is of the form

$$p(r) = \frac{p^-(R)}{2J_m(kR)}(J_m(kr) + \mathrm{i}Y_m(kr)) = \frac{p^-(R)}{2J_m(kR)}\overline{H_m^{(2)}(kr)} \tag{23}$$

Note that for a fixed $|z|$, $\mathrm{Re}(z)$ reaches its extrema for $\arg(z) = 0$ and $\arg(z) = \pi$, i.e., when z is real. Applying this observation to the expression for power (21), we conclude that power is minimized when $c = \lambda p^-(R)$ for some real λ. Minimizing with respect to λ proves the above theorem.

Once the overall exterior pressure field p after the control is known, see (23), the required control inputs themselves can be constructed using formulae (10) and (11). Namely, we have to select $w(\boldsymbol{x})$ in the definition (10) so that for $\boldsymbol{x} \in \Omega_1$ we would obtain $p^+(\boldsymbol{x}) + p^-(\boldsymbol{x}) + p^{(\text{surf})}(\boldsymbol{x}) = p(\boldsymbol{x})$, where $p^{(\text{surf})}(\boldsymbol{x})$ is given by formula (11) and the Fourier components of $p(\boldsymbol{x})$ are given by formula (23). Substitution yields: $w(\boldsymbol{x}) = 2p^+(\boldsymbol{x}) + p^-(\boldsymbol{x}) - p(\boldsymbol{x})$, and transforming into the Fourier space, we obtain for each m (subscript omitted, as before):

$$w(r) = 2p^+(r) + p^-(r) - p(r) = 2p^+(r) - cH_m^{(2)}(kr), \quad \text{for } r \geq R \tag{24}$$

where the value of c in (24) is the optimum given by formula (22). Expression (24) was obtained with the help of (20), where the first term on the right-hand side of (20) is, obviously, $p^-(r)$. Representation (24) implies that $\boldsymbol{L}w = 0$, $\boldsymbol{x} \in \Omega_1$, as it is supposed to be according to Section 2.

Using expression (24) in formula (10), where notation $w - p = w - p^- - p^+$ applies, we obtain the power optimal control inputs at each (implied) m:

$$g^{(\text{surf})}(r) = - \left.\frac{\partial[p^+(r) - p^-(r) - cH_m^{(2)}(r)]}{\partial r}\right|_{r=R} \delta(r) - \frac{\partial}{\partial r}([p^+(r) - p^-(r) - cH_m^{(2)}(r)]|_{r=R}\delta(r))$$

8. Discussion

Let us assume that p^- is due to a point source at $s > R$, so that

$$p^-(R) = -\frac{1}{4\mathrm{i}} J_m(kR) H_m^{(2)}(ks)$$

and

$$W^- = -\frac{|H_m^{(2)}(ks)|^2}{8\rho\omega}$$

This clearly shows that at each ω and m one can extract power from the noise field. However, the amount of power that can be extracted grows very rapidly as $m \to \infty$:

$$W^- \sim -\frac{\Gamma(m)^2}{8\rho\omega\pi^2}\left(\frac{2}{ks}\right)^{2m}$$

This asymptotic result need not preclude practical applications of Theorem 2 which will necessarily be limited to $|m| \leq M$ for some application dependent constant M. For $|m| > M$, one can adopt a policy of leaving the exterior field unaltered by interior sources or controls, which will result in $W^- = 0$ for $|m| > M$.

How is it possible to extract large amounts of power from the exterior acoustic field? The answer lies in the fact that optimal controls increase acoustic loading at the noise source locations. Choosing larger M allows us to increase loading on the noise sources to the point that in practice their acoustic strength [1] would decrease. Theorem 2 remains applicable, but the decreased acoustic strength of noise sources would limit the amount of power that can be extracted in practice even with large M.

What happens to the increased noise power if the acoustic strength does not decrease? The increased power is partly absorbed at the perimeter Γ and partly radiated to infinity. For the simple example of a single point source of noise one can show that half of its radiated power for each m is absorbed at the control layer and half is lost to waves radiating towards infinity.

This paper demonstrates that power optimal active noise control can be a significant net producer of power. Clearly, all power radiated by the interior sound sources can be absorbed by the controls. Optimal controls increase acoustic loading on the exterior noise sources and thereby absorb additional power limited only by the number of controlled circumferential modes and by the decrease in acoustic strength of the noise sources in response to the increased acoustic loading.

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