ACTIVE SHIELDING AND CONTROL OF NOISE[*]

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Abstract. We present a mathematical framework for the active control of time-harmonic acoustic disturbances. Unlike many existing methodologies, our approach provides for the exact volumetric cancellation of unwanted noise in a given predetermined region of space while leaving unaltered those components of the total acoustic field that are deemed friendly. Our key finding is that for eliminating the unwanted component of the acoustic field in a given area, one needs to know relatively little; in particular, neither the locations nor structure nor strength of the exterior noise sources need to be known. Likewise, there is no need to know the volumetric properties of the supporting medium across which the acoustic signals propagate, except, perhaps, in the narrow area of space near the boundary (perimeter) of the domain to be shielded. The controls are built based solely on the measurements performed on the perimeter of the region to be shielded; moreover, the controls themselves (i.e., additional sources) are also concentrated only near this perimeter. Perhaps as important, the measured quantities can refer to the total acoustic field rather than only to its unwanted component, and the methodology can automatically distinguish between the two.

In the paper, we construct a general solution to the aforementioned noise control problem. The apparatus used for deriving the general solution is closely connected to the concepts of generalized potentials and boundary projections of Calderon's type. For a given total wave field, the application of Calderon's projections allows us to definitively split its incoming and outgoing components with respect to a particular domain of interest, which may have arbitrary shape. Then the controls are designed so that they suppress the incoming component for the domain to be shielded or alternatively, the outgoing component for the domain, which is complementary to the one to be shielded. To demonstrate that the new noise control technique is appropriate, we thoroughly work out a two-dimensional model example that allows full analytical consideration.

To conclude, we very briefly discuss the numerical (finite-difference) framework for active noise control that has, in fact, already been worked out, as well as some forthcoming extensions of the current work: optimization of the solution according to different criteria that would fit different practical requirements, applicability of the new technique to quasi-stationary problems, and active shielding in the case of the broad-band spectra of disturbances. In the future, the aforementioned finite-difference framework for active noise control is going to be used for analyzing complex configurations that originate from practical designs.

Key words. active shielding, noise control, time-harmonic acoustic fields, the Helmholtz equation, generalized Calderon's potentials, exact volumetric cancellation, general solution, incoming and outgoing waves, near surface control sources, spatial anisotropies, material discontinuities, optimization, Bessel functions

AMS subject classifications. 35J05, 35C15, 31C99, 47G30, 35B37

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1. Introduction.

1.1. Background. The area of active shielding and noise control has an extensive history for a variety of applications. No adequate review of the field can be provided in the framework of a focused research publication, and the examples we mention here are simply representative. They do, however, establish the relevance of our study. Elliot, Stothers, and Nelson in [1] develop a procedure for minimizing the noise level at a number of pointwise locations (where the sensors—microphones—are actually placed); Wright and Vuksanovic in [2, 3] discuss directional noise cancellation. Various issues associated with the development and implementation of noise cancellation techniques for aircraft industry applications are analyzed in collection volumes [4, 5]. Van der Auweraer et al. in [6] studied the methods of aircraft interior noise control using both acoustical and structural excitation.

The work by Kincaid et al. in active noise control [7, 8, 9, 10, 11] has been done under the assumptions that the noise source is a well-understood monopole with a few distinct frequencies (in the 200–500 Hz range) and that the volume properties of the supporting medium are relatively well understood. The goal was to cancel out the static noise at a given collection of sensors distributed throughout the aircraft cabin by means of designing an appropriate system of actuators. In [12] and [13], Kincaid and Berger solved the actuator placement problem for the mathematically analogous case of a large flexible space structure. The goal was to place actuators so as to be able to effectively damp structural vibrations for large numbers of modes (cf. [14]).

An overview of the recent practice in active control of noise is presented in work by Fuller and von Flotow [15]. A typical noise reduction system would minimize the noise measured at multiple sensing locations by adaptively tuning the control filter. The filtered-x least mean squares (LMS) adaptation mechanism originally introduced by Burgess [16] and Widrow, Schur, and Schaffer [17] is most often employed. Another adaptation mechanism called principal component LMS algorithm (PC-LMS), presented by Cabell and Fuller in [18], offers faster convergence. Good results can be achieved at the sensor locations, but other locations are not considered in the problem formulation. By contrast, the approach proposed here offers the exact volumetric cancellation of noise.

In contrast to many existing noise control methodologies, the active shielding technique that we are proposing here suggests the possibility of *exact uniform volumetric noise cancellation*; moreover, this can be done, if necessary, through the use of only surface sensors and actuators. This paper contains *a systematic description of the mathematical foundations of the new methodology in the continuous formulation*, including a model example for the two-dimensional Helmholtz equation. The corresponding finite-difference framework has been previously introduced and studied in the series of publications by Ryaben'kii and coauthors [19, 20, 21, 22].

Our technique actually allows one to split the total acoustic field into two components, the first deemed "friendly" and called sound, the second deemed "adverse" and called noise (see sections 1.2, 3.2), and subsequently provides for the *exact volumetric cancellation of the adverse component while leaving the friendly one unaffected*. The input data for the noise control system are some quantities measured on the perimeter of the region to be shielded (see section 2). A very important feature of our approach is that *the measured quantities can refer to the total acoustic field rather than only to its adverse component.* Then, the foregoing split into sound and noise and subsequent noise cancellation is performed by the control system automatically with no special attention required.

Let us mention that the analysis tools that we employ for building our methodology involve boundary integrals of the kind routinely used in the classical potential theory (see section 2) and then extend to the operator representation and generalized boundary projection operators of Calderon's type (see section 3). The apparatus of the classical potential theory as applied to the Helmholtz equation has been used in the past for the analysis of active noise control problems; see, e.g., the monograph by Nelson and Elliot [23]. In this book, the authors construct special control sources on the surface of the region to be shielded for what they call active absorption and reflection of the undesirable noise. These sources are obtained as dipole and monopole layers (for absorption) and only monopole layers (for reflection), with the responses in the form of double- and single-layer potentials and only single-layer potentials, respectively. *The fundamental difference* between our study and that of [23] is that *the authors of* [23] *need to know the actual boundary trace of the noise field to be canceled* and construct their control sources with the explicit use of these data. In other words, their analysis in this part reduces to the solution of the boundary-value problems for the Helmholtz equation using boundary integral equations of the classical potential theory. As opposed to [23], we do not need to know the actual "adverse" component of the acoustic field that we want to cancel; this is most advantageous because this component is obviously impossible to measure directly in the presence of other acoustic signals. What is clearly possible to measure directly is the overall acoustic field that may also include a "friendly" sound component (generated by the interior sources) that is supposed to be unaffected by the controls. *This overall field is all we need to know for building the controls.* As has been mentioned, our control methodology is capable of automatically distinguishing between the two components (friendly and adverse) and accordingly responding only to the adverse part. Besides, we provide for a closed form general solution for the control sources and can analyze more complex settings compared to the original Helmholtz equation, in particular, anisotropies, material discontinuities, and certain types of nonlinearities.

Another issue to be emphasized regarding our work is somewhat counterintuitive and deviates from the "conventional wisdom." It turns out, in fact, that *one needs to know relatively little* to be able to cancel the undesirable noise; in particular, no detailed information about either structure or location or strength of the noise sources, or about the volumetric properties of the supporting medium, is required. We also stress that our methodology yields *a closed form of the general solution to the noise control problem;* availability of this general solution provides, in particular, a powerful means for optimization. Moreover, the same technique will apply to media with *different acoustic conductivities and inhomogeneities* (within the linear regime), as both the adverse noise that we need to cancel and the output of the control system propagate across the same medium. Finally, the methodology can be applied to *quasistationary regimes and, in the future, to broad-band spectra of frequencies.*

1.2. Formulation of the problem. Let Ω be a given domain, $\Omega \subset \mathbb{R}^n$, $n = 2$ or $n = 3$ (higher dimensions n can be analyzed similarly; we, however, restrict ourselves to the cases that originate from physical applications). The domain Ω can be either bounded or unbounded; in the beginning we will assume for simplicity that Ω is bounded. Let Γ be the boundary of Ω: $\Gamma = \partial\Omega$. Both on Ω and on its complement $\mathbb{R}^n \backslash \Omega$ we consider the time-harmonic acoustic field u governed by the nonhomogeneous Helmholtz equation

(1.1) $$\boldsymbol{L}u \equiv \Delta u + k^2 u = f.$$
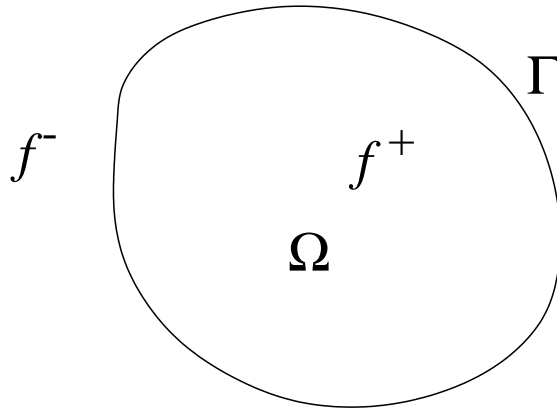
Fig. 1.1. *Geometric setup.*

The sources f in (1.1) can be located on both Ω and its complement $\mathbb{R}^n \backslash \Omega$; to emphasize the distinction, we denote

$$(1.2) \qquad\qquad f = f^+ + f^-,$$

where the sources f^+ are interior, $\operatorname{supp} f^+ \subset \Omega$, and the sources f^- are exterior, $\operatorname{supp} f^- \subset \mathbb{R}^n \backslash \Omega$. Accordingly, the overall acoustic field u can be represented as a sum of two components:

$$(1.3) \qquad\qquad u = u^+ + u^-,$$

where

$$(1.4a) \qquad\qquad \boldsymbol{L} u^+ = f^+,$$
$$(1.4b) \qquad\qquad \boldsymbol{L} u^- = f^-.$$

Note that both u^+ and u^- are defined on the entire $\mathbb{R}^n$, and the superscripts "+" and "−" refer to the sources that drive each of the field components rather than to the domains of these components. The setup described above is schematically shown in Figure 1.1.

Hereafter, we will call the component u^+ of (1.3) (see (1.4a)) *sound* or the "friendly" part of the total acoustic field; the component u^- of (1.3) (see (1.4b)) will be called *noise* or the "adverse" part of the total acoustic field. In the formulation that we are presenting, Ω will be a (predetermined) region of space to be shielded. This means that we would like to eliminate the noise inside Ω while leaving the sound component there unaltered. In the mathematical framework that we have adopted, the component u^- of the total acoustic field, i.e., the response to the adverse sources f^- (see (1.2), (1.3), (1.4)), will have to be canceled on Ω, whereas the component u^+, i.e., the response to the friendly sources f^+, will have to be left intact on Ω. A physically more complex but conceptually easy to understand example that can be given is that inside the passenger compartment of an aircraft we would like to eliminate the noise coming from the propulsion system located outside the aircraft fuselage while not interfering with the ability of the passengers to listen to the in-flight entertainment programs or simply converse.

The concept of *active shielding* (noise cancellation) that we will be discussing implies that the aforementioned goal is to be achieved by introducing additional sources g, $\operatorname{supp} g \subset \mathbb{R}^n \backslash \Omega$, so that the total acoustic field $\tilde{u}$ will now be governed by the equation

$$(1.5) \qquad \qquad \boldsymbol{L}\tilde{u} = f^+ + f^- + g$$

and coincide only with the sound component u^+ on the domain Ω:

$$(1.6) \qquad \qquad \tilde{u}\Big|_{x \in \Omega} = u^+\Big|_{x \in \Omega},$$

$x = (x_1, x_2)$ for $n = 2$, and $x = (x_1, x_2, x_3)$ for $n = 3$.

The new sources g of (1.5) will hereafter be referred to as *controls*. Let us note that in practical settings, when one typically has to deal with acoustic fields composed of signals with multiple frequencies rather than time-harmonic single-frequency fields, active strategies of noise reduction are often combined with passive ones. Passive strategies that employ different types of sound insulation have proven efficient for damping the high-frequency signals, while low-frequency noise components more easily lend themselves to reduction by active techniques. We also note that the words "noise cancellation" should not be interpreted incorrectly. As will be seen from the forthcoming analysis, elimination of the component u^- inside Ω also implies changing the total acoustic field outside Ω. Therefore, in terms of the acoustic energy, it means *redistribution rather than cancellation*.

An obvious solution to the foregoing noise control problem is $g = -f^-$. As, however, will become clear, this solution is excessively expensive. On one hand, this expensiveness relates to the informational considerations as the solution $g = -f^-$ requires an explicit and detailed knowledge of the structure and location of the sources f^-, which, in fact, will prove superfluous. On the other hand, the implementation of this solution may encounter most serious difficulties. In the previous example associated with an aircraft, it is obviously not feasible to directly counter the noise sources which are aircraft propellers or turbofan jet engines located on or underneath the wings. Therefore, other solutions of the control problem, besides the most obvious one, may be preferable from both theoretical and practical standpoints. The general solution of the control problem formulated in this section is constructed in the forthcoming section 2.

2. General solution of the control problem. Let us first introduce fundamental solutions to the Helmholtz operator. By definition, these are solutions to the nonhomogeneous equation (1.1) driven by the δ-source located at the origin. For $n = 2$ we have

$$(2.1a) \qquad \qquad G(x) = -\frac{1}{4i} H_0^{(2)}(k|x|),$$

where $H_0^{(2)}(z)$ is the Hankel function of the second kind defined by means of the Bessel functions $J_0(z)$ and $Y_0(z)$ as $H_0^{(2)}(z) = J_0(z) - iY_0(z)$. For $n = 3$

$$(2.1b) \qquad \qquad G(x) = -\frac{e^{-ik|x|}}{4\pi|x|}.$$

Let us note that both solutions (2.1) satisfy the so-called Sommerfeld radiation condition at infinity; see, e.g., [24, 25]. This condition, which for a given function $u(x)$ is

formulated as

$$(2.2a) \qquad u(x) = O\left(|x|^{-1/2}\right), \quad \frac{\partial u(x)}{\partial |x|} + iku(x) = o\left(|x|^{-1/2}\right) \quad \text{as} \quad |x| \longrightarrow \infty$$

for two space dimensions and

$$(2.2b) \qquad u(x) = O\left(|x|^{-1}\right), \quad \frac{\partial u(x)}{\partial |x|} + iku(x) = o\left(|x|^{-1}\right) \quad \text{as} \quad |x| \longrightarrow \infty$$

for three space dimensions, specifies the direction of wave propagation and essentially distinguishes between the incoming and outgoing waves at infinity. The Sommerfeld condition (2.2a) or (2.2b) is required to guarantee uniqueness of the solution to the Helmholtz equation; unlike for the Laplace equation, the condition of boundedness ($n = 2$) or vanishing ($n = 3$) at infinity is not sufficient for the solution of the Helmholtz equation to be unique. Provided that the following convolution exists in some sense, the solution $u(x)$ to the nonhomogeneous equation (1.1) defined on $\mathbb{R}^n$ and satisfying the Sommerfeld condition is given by

$$(2.3) \qquad u(x) = \int_{\mathbb{R}^n} f(y)G(x-y)dy.$$

Next, according to the standard procedure (see, e.g., [24]), we construct the surface and volume potentials for the Helmholtz operator and write the Green's formula

$$(2.4) \qquad u(x) = \underbrace{\int_\Omega G \boldsymbol{L} u \, dy}_{\text{volume potential}} + \underbrace{\int_\Gamma \left(u \frac{\partial G}{\partial \boldsymbol{n}} - \frac{\partial u}{\partial \boldsymbol{n}} G \right) ds_y}_{\text{two surface potentials}}, \quad x \in \Omega,$$

which holds for any sufficiently smooth function $u(x)$. All integrals on the right-hand side of (2.4) are, of course, convolutions, and $\boldsymbol{n}$ is the outward normal to the boundary Γ. The second component on the right-hand side of (2.4) is the sum of two surface potentials and thus it satisfies the homogeneous Helmholtz equation on Ω:

$$(2.5) \qquad \boldsymbol{L} \int_\Gamma \left(u \frac{\partial G}{\partial \boldsymbol{n}} - \frac{\partial u}{\partial \boldsymbol{n}} G \right) ds_y = 0, \quad x \in \Omega.$$

The sound component u^+ of the total acoustic field satisfies (1.4a) on $\mathbb{R}^n$, and as the Sommerfeld condition is built into the structure of $G(x)$ (see (2.1)), the function $u^+(x)$ can be obtained as follows:

$$(2.6) \qquad u^+(x) = \int_\Omega G f^+ dy = \int_\Omega G \boldsymbol{L} u^+ dy, \quad x \in \mathbb{R}^n.$$

Then, the application of the Green's formula (2.4) to u^+ of (2.6) immediately yields

$$(2.7) \qquad \int_\Gamma \left(u^+ \frac{\partial G}{\partial \boldsymbol{n}} - \frac{\partial u^+}{\partial \boldsymbol{n}} G \right) ds_y = 0, \quad x \in \Omega.$$

Let us now recall that according to (1.2) and (1.4b) for $x \in \Omega$ there will be

$$(2.8) \qquad \boldsymbol{L} u^- \big|_{x \in \Omega} = 0.$$

Thus, applying the Green's formula (2.4) to the total acoustic field $u = u^+ + u^-$ and using relations (2.7) and (2.8) we obtain

$$u(x) = \int_\Omega G\boldsymbol{L}(u^+ + u^-)dy + \int_\Gamma \left((u^+ + u^-)\frac{\partial G}{\partial \boldsymbol{n}} - \frac{\partial (u^+ + u^-)}{\partial \boldsymbol{n}}G \right) ds_y$$

(2.9)

$$= \underbrace{\int_\Omega G\boldsymbol{L}u^+ dy}_{u^+} + \underbrace{\int_\Gamma \left(u^-\frac{\partial G}{\partial \boldsymbol{n}} - \frac{\partial u^-}{\partial \boldsymbol{n}}G \right) ds_y}_{u^-}, \quad x \in \Omega.$$

Therefore, if we define the annihilating function $v(x)$, $x \in \Omega$,

(2.10) $$v(x) = -\int_\Gamma \left(u^-\frac{\partial G}{\partial \boldsymbol{n}} - \frac{\partial u^-}{\partial \boldsymbol{n}}G \right) ds_y = -u^-(x),$$

then inside Ω we will have

(2.11) $$u + v\Big|_{x\in\Omega} = u^+.$$

Consequently, if, by some means, we obtain the two quantities u^- and $\frac{\partial u^-}{\partial \boldsymbol{n}}$ at the boundary Γ, then we can produce the annihilating function v by formula (2.10) and cancel the adverse component u^- of the acoustic field inside Ω. In practical settings, the quantities u^- and $\frac{\partial u^-}{\partial \boldsymbol{n}}$ or similar ones should be actually measured.

A remarkable property of the function $v(x)$ defined by formula (2.10) is that by virtue of (2.7)

(2.12) $$v(x) = -\int_\Gamma \left(u^-\frac{\partial G}{\partial \boldsymbol{n}} - \frac{\partial u^-}{\partial \boldsymbol{n}}G \right) ds_y = -\int_\Gamma \left(u\frac{\partial G}{\partial \boldsymbol{n}} - \frac{\partial u}{\partial \boldsymbol{n}}G \right) ds_y, \quad x \in \Omega.$$

Thus, *the quantities to be measured at the boundary* Γ, u *and* $\frac{\partial u}{\partial \boldsymbol{n}}$, *may refer to the total field* $u = u^+ + u^-$ *rather than its adverse component* u^- *only*. The annihilating function v defined as

(2.13) $$v(x) = -\int_\Gamma \left(u\frac{\partial G}{\partial \boldsymbol{n}} - \frac{\partial u}{\partial \boldsymbol{n}}G \right) ds_y, \quad x \in \Omega,$$

automatically filters out the contribution from the friendly sound u^+ and responds only to the adverse noise component u^-.

Formula (2.13) that defines the annihilating function $v(x)$ contains surface integrals of the quantities u and $\frac{\partial u}{\partial \boldsymbol{n}}$. We will now discuss the alternative, more apparent, means of building $v(x)$. Consider a sufficiently smooth function $w(x)$ defined on $\mathbb{R}^n$ that satisfies the Sommerfeld condition (2.2a) or (2.2b) and additional boundary conditions on Γ:

(2.14) $$w\Big|_\Gamma = u\Big|_\Gamma, \qquad \frac{\partial w}{\partial \boldsymbol{n}}\Big|_\Gamma = \frac{\partial u}{\partial \boldsymbol{n}}\Big|_\Gamma.$$

Here u is the actual total acoustic field, as before. In particular, $w(x)$ may be compactly supported near the boundary Γ. Obviously (because of (2.14)),

(2.15) $$v(x) = -\int_\Gamma \left(u\frac{\partial G}{\partial \boldsymbol{n}} - \frac{\partial u}{\partial \boldsymbol{n}}G \right) ds_y = -\int_\Gamma \left(w\frac{\partial G}{\partial \boldsymbol{n}} - \frac{\partial w}{\partial \boldsymbol{n}}G \right) ds_y, \quad x \in \Omega.$$

On the other hand, if we apply the Green's formula (2.4) to the function w, we obtain

$$(2.16) \qquad w(x) - \int_\Omega G\boldsymbol{L}w\,dy = \int_\Gamma \left(w\frac{\partial G}{\partial \boldsymbol{n}} - \frac{\partial w}{\partial \boldsymbol{n}}G \right) ds_y, \quad x \in \Omega.$$

An important consideration now is to remember that the Sommerfeld boundary condition (2.2a) or (2.2b) guarantees uniqueness, and therefore

$$(2.17) \qquad w(x) = \int_{\mathbb{R}^n} G\boldsymbol{L}w\,dy, \quad x \in \mathbb{R}^n.$$

If $w(x)$ is compactly supported, then integration in (2.17) can actually be performed over $\operatorname{supp} w$ rather than entire $\mathbb{R}^n$. Relations (2.16) and (2.17) together imply that

$$(2.18) \qquad \int_{\mathbb{R}^n \setminus \Omega} G\boldsymbol{L}w\,dy = \int_\Gamma \left(w\frac{\partial G}{\partial \boldsymbol{n}} - \frac{\partial w}{\partial \boldsymbol{n}}G \right) ds_y, \quad x \in \Omega.$$

Therefore,

$$(2.19) \qquad v(x) = -\int_{\mathbb{R}^n \setminus \Omega} G\boldsymbol{L}w\,dy = \int_{\mathbb{R}^n \setminus \Omega} Gg\,dy,$$

where the control g in (2.19) is defined as

$$(2.20) \qquad g(x) = -\boldsymbol{L}w\Big|_{x \in \mathbb{R}^n \setminus \Omega}$$

provided that w satisfies (2.2a) or (2.2b) and (2.14). A variety of choices for w (under the conditions (2.2a) or (2.2b) and (2.14)) implies that there is also a variety of controls g (see (2.20)) that solve the problem; this, in particular, *gives room for optimization.*

PROPOSITION 2.1. *Formula* (2.20) *describes the entire variety of the appropriate controls* g.

Proof. We have already seen that formula (2.20) does give a solution to the active shielding problem. It remains to demonstrate that any solution to this problem can be represented in the form (2.20). Consider a control function g, $\operatorname{supp} g \in \mathbb{R}^n \setminus \Omega$, such that the solution $\tilde{u}$ of (1.5) satisfies the Sommerfeld condition (2.2a) or (2.2b) and also equality (1.6). We need to find w that meets conditions (2.2a) or (2.2b) and (2.14) and such that g can be calculated by formula (2.20). We introduce $w(x)$, $x \in \mathbb{R}^n$, as the solution to the nonhomogeneous equation

$$(2.21) \qquad -\boldsymbol{L}w = g - f^+$$

subject to the corresponding Sommerfeld condition, (2.2a) or (2.2b). Equality (1.6) for the solution $\tilde{u}$ of (1.5) implies that the function $\tilde{w}$ that solves equation $\boldsymbol{L}\tilde{w} = g$, subject to the same Sommerfeld condition, coincides with $-u^-$ on Ω: $\tilde{w}\big|_\Omega = -u^-\big|_\Omega$. Consequently, the function w introduced above satisfies

$$(2.22) \qquad w\Big|_{x \in \Omega} = u^- + u^+\Big|_{x \in \Omega} = u\Big|_{x \in \Omega},$$

and therefore, it satisfies conditions (2.14) as well. Since $\operatorname{supp} f^+ \subset \Omega$, the control g can be obtained by formula (2.20). □

3. Connection to the theory of Calderon's potentials. The foregoing split between u^+ and u^- (see formulae (2.12), (2.13)) can be conveniently described in terms of *the generalized potentials and boundary projection operators of Calderon's type.* We refer here to the original work by Calderon [26], followed by the paper by Seeley [27] and then work by Ryaben'kii [28, 29, 30], in which the actual form of the operators used, in particular, in this paper was introduced; a brief account of Ryaben'kii's work can also be found in the book by Mikhlin, Morozov, and Paukshto [31].

3.1. Potential and projection for the domain Ω. Consider some function $u(x)$ such that $\boldsymbol{L}u = 0$ for $x \in \Omega$. Then, the Green's formula (2.4) yields

$$(3.1) \qquad u(x) = \int_\Gamma \left(u \frac{\partial G}{\partial \boldsymbol{n}} - \frac{\partial u}{\partial \boldsymbol{n}} G \right) ds_y, \quad x \in \Omega.$$

We emphasize that the representation (3.1) of a function u as a sum of a double-layer potential with the density $u\big|_\Gamma$ and a single-layer potential with the density $\frac{\partial u}{\partial \boldsymbol{n}}\big|_\Gamma$ is valid only for the functions that solve the homogeneous equation $\boldsymbol{L}u = 0$ on the domain Ω. If, however, we specify two arbitrary functions on Γ and substitute them into (3.1) as densities of the potentials, then the resulting function will obviously be a solution of $\boldsymbol{L}u = 0$ on Ω, but its boundary values, as well as boundary values of its normal derivative, will not, generally speaking, coincide with the original densities of the double-layer and single-layer potentials, respectively. A generalized potential of Calderon's type with the vector density $\xi_\Gamma = (\xi_0, \xi_1)$ specified on Γ is defined by the following formula:

$$(3.2) \qquad \boldsymbol{P}_\Omega \xi_\Gamma(x) = \int_\Gamma \left(\xi_0 \frac{\partial G}{\partial \boldsymbol{n}} - \xi_1 G \right) ds_y, \quad x \in \Omega,$$

which is similar to (3.1) except that we do not require ahead of time that ξ_0 and ξ_1 in (3.2) be the boundary values of some function that solves $\boldsymbol{L}u = 0$ on Ω. If, on the other hand, ξ_0 and ξ_1 are the boundary values of u and $\frac{\partial u}{\partial \boldsymbol{n}}$ for some solution to $\boldsymbol{L}u = 0$ on Ω, then the Green's formula (3.1) for this u can be written in a shortened form:

$$(3.3) \qquad u = \boldsymbol{P}_\Omega \left(u, \frac{\partial u}{\partial \boldsymbol{n}} \right)\bigg|_\Gamma, \quad x \in \Omega.$$

For any (sufficiently smooth) function v specified on Ω we also define its vector trace on Γ,

$$(3.4) \qquad \boldsymbol{Tr}\, v = \left(v, \frac{\partial v}{\partial \boldsymbol{n}} \right)\bigg|_\Gamma,$$

and then introduce the boundary operator $\boldsymbol{P}_\Gamma$ as a combination of the potential $\boldsymbol{P}_\Omega$ of (3.2) and trace $\boldsymbol{Tr}$ of (3.4):

$$(3.5) \qquad \boldsymbol{P}_\Gamma \xi_\Gamma = \boldsymbol{Tr}\, \boldsymbol{P}_\Omega \xi_\Gamma.$$

Clearly, the operator $\boldsymbol{P}_\Gamma$ of (3.5) is a projection, $\boldsymbol{P}_\Gamma = \boldsymbol{P}_\Gamma^2$. Indeed, $\forall \xi_\Gamma : \boldsymbol{L}\boldsymbol{P}_\Omega \xi_\Gamma = 0$, $x \in \Omega$. Therefore, the Green's formula (3.3) yields $\boldsymbol{P}_\Omega \xi_\Gamma = \boldsymbol{P}_\Omega \boldsymbol{Tr} \boldsymbol{P}_\Omega \xi_\Gamma$, which immediately implies that $\boldsymbol{P}_\Gamma$ is a projection.

The key property of the operator $\boldsymbol{P}_\Gamma$ of (3.5) is the following.

PROPOSITION 3.1. *Those and only those vector-functions ξ_Γ that satisfy the boundary equation with projection (BEP)*

$$(3.6) \qquad\qquad \boldsymbol{P}_\Gamma \xi_\Gamma = \xi_\Gamma$$

can be complemented on Ω to a function u such that $\boldsymbol{L}u = 0$ on Ω and $\boldsymbol{Tr}\, u = \xi_\Gamma$.

In other words, the range $\mathrm{Im}\boldsymbol{P}_\Gamma$ of the projection operator $\boldsymbol{P}_\Gamma$ given by (3.5) exhaustively characterizes all those and only those boundary functions that admit a complement to the domain Ω in the foregoing sense.

Proof. Let u be a sufficiently smooth function defined on Ω, $\boldsymbol{L}u = 0$. Then applying the operator $\boldsymbol{Tr}$ of (3.4) to the Green's representation (3.3) for u, we obtain the BEP (3.6). Conversely, let equality (3.6) hold for some ξ_Γ. Denote $u = \boldsymbol{P}_\Omega \xi_\Gamma$; obviously $\boldsymbol{L}u = 0$ on Ω. In addition, equality (3.6) implies that $\boldsymbol{Tr}\, u = \xi_\Gamma$. Thus, we have obtained the required complement.   ☐

Note, although we have described the Calderon potentials and projections using the language of surface integrals, the previously introduced constructions that use the auxiliary function w—in particular, formula (2.16)—will apply here with no change. We emphasize that formula (2.16) essentially shows how to replace surface integrals by volume integrals in the entire derivation. In particular, given a vector density $\xi_\Gamma = (\xi_0, \xi_1)$, we can take a sufficiently smooth function $w(x)$ compactly supported near Γ such that (cf. (2.14))

$$(3.7) \qquad\qquad \boldsymbol{Tr}\, w = \xi_\Gamma$$

and redefine the potential (3.2) as follows:

$$(3.8)
\begin{aligned}
\boldsymbol{P}_\Omega \xi_\Gamma(x) &= w(x) - \int_\Omega G\boldsymbol{L}w\, dy \\
&= \int_{\mathbb{R}^n \setminus \Omega} G\boldsymbol{L}w\, dy, \qquad x \in \Omega.
\end{aligned}$$

Obviously, definitions (3.2) and (3.8) are equivalent provided that relations (3.7) hold. The second equality in (3.8) holds because of the uniqueness in the form (2.17). Let us emphasize that *the potential $\boldsymbol{P}_\Omega \xi_\Gamma$ obviously does not depend on the specific choice of the auxiliary function $w(x)$ as long as condition (3.7) is met.*

3.2. Split between u^+ and u^-. Incoming and outgoing waves. We now return to formulae (2.10)–(2.13). For the annihilating function $v(x)$, $x \in \Omega$, we have $v = -u^-$, $x \in \Omega$. Equation (1.4b), along with the consideration that $\mathrm{supp}\, f^- \subset \mathbb{R}^n \setminus \Omega$, implies $\boldsymbol{L}u^- = 0$, $x \in \Omega$. Therefore, if we denote $\xi_\Gamma^- = (u^-, \frac{\partial u^-}{\partial n})|_\Gamma$, then, obviously, ξ_Γ^- satisfies BEP (3.6), $\boldsymbol{P}_\Gamma \xi_\Gamma^- = \xi_\Gamma^-$, i.e., $\xi_\Gamma^- \in \mathrm{Im}\, \boldsymbol{P}_\Gamma$, and relation (2.10) becomes

$$(3.9) \qquad\qquad v = -\boldsymbol{P}_\Omega \xi_\Gamma^-.$$

Let us also denote $\xi_\Gamma = (u, \frac{\partial u}{\partial n})|_\Gamma$; here again u is the overall acoustic field, as in section 2. Then, we can rewrite formula (2.12) as follows:

$$(3.10) \qquad\qquad v = -\boldsymbol{P}_\Omega \xi_\Gamma^- = -\boldsymbol{P}_\Omega \xi_\Gamma, \qquad x \in \Omega.$$

Thus, the annihilating function $v(x)$, $x \in \Omega$, can be obtained as a generalized Calderon's potential with the density $-\xi_\Gamma = -(u, \frac{\partial u}{\partial n})|_\Gamma$, and the second equality of

(3.8) *again indicates that $v(x)$ can actually be calculated through the use of the auxiliary function $w(x)$ that satisfies boundary condition* (3.7). Moreover, as the potential $\boldsymbol{P}_\Omega \xi_\Gamma$ depends only on ξ_Γ and not on the particular choice of $w(x)$, we in fact obtain the entire variety of controls (2.20) that cancel out the same acoustic disturbance $u^-(x)$ on the domain Ω.

Next, applying the operator $\boldsymbol{Tr}$ of (3.4) to both sides of the second equality in (3.10) and taking into account that $\boldsymbol{P}_\Gamma \xi_\Gamma^- = \xi_\Gamma^-$, we obtain

$$(3.11) \qquad\qquad \boldsymbol{P}_\Gamma \xi_\Gamma = \xi_\Gamma^-.$$

Equality (3.11) means that the boundary function ξ_Γ, which is the vector trace (in the sense of (3.4)) of the total acoustic field $u(x)$, does not, generally speaking, satisfy the BEP (3.6) (unless $\xi_\Gamma^- = \xi_\Gamma \iff \xi_\Gamma^+ \equiv (u^+, \frac{\partial u^+}{\partial \boldsymbol{n}})|_\Gamma = 0$) and thus cannot, generally speaking, be complemented on Ω to a solution of the homogeneous equation $\boldsymbol{L}u = 0$. The portion of ξ_Γ that does admit the interior complement is ξ_Γ^-. What in fact happens is that the projection $\boldsymbol{P}_\Gamma$ selects only that portion of ξ_Γ that can be complemented on Ω, and the corresponding complement taken with the minus sign provides the annihilating function v according to (3.9). *Equality* (3.10) *implies that the generalized potential $\boldsymbol{P}_\Omega$ is insensitive to any contribution to ξ_Γ that does not belong to the range* $\mathrm{Im}\, \boldsymbol{P}_\Gamma$ *of the boundary projection* (3.5).

The foregoing discussion implies that the split of the total acoustic field $u(x)$ into $u^+(x)$ and $u^-(x)$ rendered by formulae (2.12), (2.13) (see also (3.10)) is in fact realized through the split of the boundary trace $\xi_\Gamma = (u, \frac{\partial u}{\partial \boldsymbol{n}})|_\Gamma$ into $\xi_\Gamma^+ = (u^+, \frac{\partial u^+}{\partial \boldsymbol{n}})|_\Gamma$ and $\xi_\Gamma^- = (u^-, \frac{\partial u^-}{\partial \boldsymbol{n}})|_\Gamma$:

$$(3.12) \qquad\qquad \xi_\Gamma = \xi_\Gamma^+ + \xi_\Gamma^-.$$

The component ξ_Γ^- in (3.12) is obtained by applying the projection $\boldsymbol{P}_\Gamma$ to ξ_Γ according to formula (3.11), and ξ_Γ^+ of (3.12) complements ξ_Γ^- to the entire ξ_Γ.

Assume now that all we know is the trace ξ_Γ (in the sense of (3.4)) on the boundary $\Gamma = \partial\Omega$ of some solution u (defined on $\mathbb{R}^n$) to the equation $\boldsymbol{L}u = f$ subject to the corresponding Sommerfeld condition (2.2a) or (2.2b). We do not know anything about the sources f except that they guarantee existence of the solution in the class of functions satisfying the Sommerfeld condition. Then we perform the split (3.12) and see that ξ_Γ^- can be complemented on Ω to the solution of the *homogeneous equation* $\boldsymbol{L}u^- = 0$ (because $\boldsymbol{P}_\Gamma \xi_\Gamma^- = \xi_\Gamma^-$). Thus, this component should be interpreted as the one driven only by *the exterior sources* (with respect to Ω). The component ξ_Γ^+ in (3.12) *cannot be complemented on Ω* to the solution of the homogeneous equation according to Proposition 3.1, as formula (3.11) implies that $\boldsymbol{P}_\Gamma \xi_\Gamma^+ = 0$. Therefore, it satisfies a nonhomogeneous rather than homogeneous differential equation on Ω and thus should be interpreted as the component driven by *the interior sources.* It is easy to see that ξ_Γ^+ of (3.12) is driven *only* by the interior sources because the contribution of all exterior sources to ξ_Γ is already taken into account by ξ_Γ^-. Consequently, ξ_Γ^+ of (3.12) can be complemented to the solution of the homogeneous equation outside Ω, i.e., on the domain $\mathbb{R}^n \backslash \Omega$. This exterior complement will, in addition, satisfy the Sommerfeld condition (2.2a) or (2.2b).

Let us note that there is also a straightforward way of showing that ξ_Γ^+ of (3.12), for which $\boldsymbol{P}_\Gamma \xi_\Gamma^+ = 0$, can be complemented on $\mathbb{R}^n \backslash \Omega$ in the sense mentioned above. This is done through directly constructing the Calderon's potential and projection for the complementary domain $\mathbb{R}^n \backslash \Omega$; we postpone the corresponding derivation to section 3.3.

Thus, we have seen that the function ξ_Γ^- of (3.12) that belongs to the range of the projection operator $\boldsymbol{P}_\Gamma$, $\xi_\Gamma^- \in \operatorname{Im} \boldsymbol{P}_\Gamma \Longleftrightarrow \xi_\Gamma^- = \boldsymbol{P}_\Gamma \xi_\Gamma^-$, represents the part of the total trace ξ_Γ that is accounted for by the sources outside Ω, and the part ξ_Γ^+ of (3.12) that belongs to the kernel of the projection operator $\boldsymbol{P}_\Gamma$, $\xi_\Gamma^+ \in \operatorname{Ker} \boldsymbol{P}_\Gamma \Longleftrightarrow \boldsymbol{P}_\Gamma \xi_\Gamma^+ = 0$, is accounted for by the sources inside Ω. *In other words, with respect to the particular domain* Ω, *the components* ξ_Γ^- *and* ξ_Γ^+ *(see* (3.12)*) represent boundary traces of the incoming and outgoing portions, respectively, of the total acoustic field.*

In fact, the entire space Ξ_Γ of vector-functions $\xi_\Gamma = (\xi_0, \xi_1)$ defined on $\Gamma = \partial\Omega$ can be split into the direct sum of the range and kernel of the projection operator $\boldsymbol{P}_\Gamma$:

$$(3.13) \qquad\qquad \Xi_\Gamma = \operatorname{Im}\boldsymbol{P}_\Gamma \oplus \operatorname{Ker}\boldsymbol{P}_\Gamma.$$

Equality (3.13) means that for any given $\xi_\Gamma \in \Xi_\Gamma$ there is always a unique representation in the form (3.12). Again, the physical interpretation of equality (3.13) is that *any given acoustic field can be split into the incoming and outgoing components with respect to a given domain* Ω. *This split can be performed on the boundary only by applying the Calderon's projection* $\boldsymbol{P}_\Gamma$ *to the trace of the total acoustic field.*

To conclude this section, let us emphasize only a useful and nontrivial consequence of formulae (2.14) and (2.20). Formula (2.20) describes the variety of controls $g(x)$ through the flexibility in choosing the auxiliary function $w(x)$, which should still, however, satisfy boundary conditions (2.14). Equalities (2.14) can be rewritten as (cf. (3.7))

$$(3.14) \qquad\qquad \boldsymbol{Tr}\, w = \left. \left(u, \frac{\partial u}{\partial \boldsymbol{n}} \right) \right|_\Gamma .$$

Clearly, if we add some $\tilde{\xi}_\Gamma \equiv (\tilde{\xi}_0, \tilde{\xi}_1) \in \operatorname{Ker}\boldsymbol{P}_\Gamma$ to $(u, \frac{\partial u}{\partial \boldsymbol{n}})|_\Gamma$ on the right-hand side of equality (3.14), then we will recover *a different control function* $g(x)$ but still obtain *the same annihilating function* $v(x)$, because $v(x) = -\boldsymbol{P}_\Omega[(u, \frac{\partial u}{\partial \boldsymbol{n}})|_\Gamma + \tilde{\xi}_\Gamma]$, and the potential $\boldsymbol{P}_\Omega$ is insensitive to any $\tilde{\xi}_\Gamma \in \operatorname{Ker} \boldsymbol{P}_\Gamma$. In terms of classical potential theory, i.e., Green's formulae used in section 2, adding a $\tilde{\xi}_\Gamma \in \operatorname{Ker}\boldsymbol{P}_\Gamma$ to the right-hand side of (3.14) translates into adding another, artificial, contribution of the friendly sound type $(u^+, \frac{\partial u^+}{\partial \boldsymbol{n}})|_\Gamma$ to the set of measured quantities on Γ. Formula (2.12) indicates that it does not change the annihilating function $v(x)$, although it may, of course, change the actual control input $g(x)$. According to Proposition 2.1, formula (2.20) provides a general solution to the active shielding problem. Therefore, the aforementioned alteration, i.e., addition of $\tilde{\xi}_\Gamma$, *cannot, of course, add any new elements to the overall set of control functions* g. However, it provides an alternative way of describing (i.e., parameterizing) some of the controls, which may appear helpful for optimization, especially in the discrete framework.

3.3. Potential and projection for the complementary domain $\mathbb{R}^n \backslash \Omega$. The Calderon's potential $\boldsymbol{P}_\Omega$ (see (3.8), (3.7)) and projection $\boldsymbol{P}_\Gamma$ (see (3.5)) are introduced for the interior domain Ω. In a very similar way, one can introduce the Calderon's potential and projection for the exterior domain $\mathbb{R}^n \backslash \Omega$. Given a vector density $\xi_\Gamma = (\xi_0, \xi_1)$, we again take a sufficiently smooth auxiliary function $w(x)$ compactly supported near Γ and such that it satisfies boundary conditions (3.7), and define the exterior potential $\boldsymbol{Q}_{\mathbb{R}^n \backslash \Omega}\, \xi_\Gamma(x)$, $x \in \mathbb{R}^n \backslash \Omega$, as follows (cf. (3.8)):

$$\boldsymbol{Q}_{\mathbb{R}^n\setminus\Omega}\,\xi_\Gamma(x) = w(x) - \int_{\mathbb{R}^n\setminus\Omega} G\boldsymbol{L}w\,dy$$

(3.15)

$$= \int_\Omega G\boldsymbol{L}w\,dy, \qquad x \in \mathbb{R}^n\setminus\Omega.$$

Again, similar to (3.8) the second equality in (3.15) holds due to the uniqueness in the sense of (2.17), and the potential $\boldsymbol{Q}_{\mathbb{R}^n\setminus\Omega}\,\xi_\Gamma$ does not depend on the particular choice of $w(x)$ as long as condition (3.7) holds. The corresponding underlying construction that uses the Green's formula for obtaining the potential $\boldsymbol{Q}_{\mathbb{R}^n\setminus\Omega}$ of (3.15) will be almost identical to the one that we have used for obtaining the interior potential $\boldsymbol{P}_\Omega$ (see (3.2)) except that the Green's formula similar to (3.1) will have to be written for the exterior domain $\mathbb{R}^n\setminus\Omega$ as well. The exterior projection operator $\boldsymbol{Q}_\Gamma$, $\boldsymbol{Q}_\Gamma^2 = \boldsymbol{Q}_\Gamma$, is obtained by applying the trace $\boldsymbol{Tr}$ given in (3.4) to the potential $\boldsymbol{Q}_{\mathbb{R}^n\setminus\Omega}$ given in (3.15):

(3.16) $$\boldsymbol{Q}_\Gamma\xi_\Gamma = \boldsymbol{Tr}\,\boldsymbol{Q}_{\mathbb{R}^n\setminus\Omega}\xi_\Gamma.$$

Similar to Proposition 3.1, the range $\operatorname{Im}\boldsymbol{Q}_\Gamma$ of the projection operator $\boldsymbol{Q}_\Gamma$ (see (3.16)) contains those and only those functions ξ_Γ that can be complemented on the exterior domain $\mathbb{R}^n\setminus\Omega$ to a solution of the homogeneous equation $\boldsymbol{L}u = 0$ that would also satisfy the corresponding Sommerfeld boundary condition (2.2a) or (2.2b). Comparing the definitions (3.8) and (3.5) of the potential $\boldsymbol{P}_\Omega$ and projection $\boldsymbol{P}_\Gamma$ for the domain Ω with the definitions (3.15) and (3.16) of the potential $\boldsymbol{Q}_{\mathbb{R}^n\setminus\Omega}$ and projection $\boldsymbol{Q}_\Gamma$ for the complementary domain $\mathbb{R}^n\setminus\Omega$, we immediately see that $\boldsymbol{Q}_\Gamma = \boldsymbol{I} - \boldsymbol{P}_\Gamma$, and consequently $\operatorname{Im}\boldsymbol{P}_\Gamma = \operatorname{Ker}\boldsymbol{Q}_\Gamma$ and $\operatorname{Im}\boldsymbol{Q}_\Gamma = \operatorname{Ker}\boldsymbol{P}_\Gamma$. Therefore, any boundary function from $\operatorname{Ker}\boldsymbol{P}_\Gamma$ can indeed be complemented on the exterior domain $\mathbb{R}^n\setminus\Omega$ to the solution of the homogeneous differential equation (as has been mentioned in section 3.2).

Similar to the split (3.13) we can also obtain

(3.17) $$\Xi_\Gamma = \operatorname{Im}\boldsymbol{Q}_\Gamma \oplus \operatorname{Ker}\boldsymbol{Q}_\Gamma.$$

The functions from $\operatorname{Im}\boldsymbol{Q}_\Gamma$ and $\operatorname{Ker}\boldsymbol{Q}_\Gamma$ are boundary traces (in the sense of (3.4)) of the incoming and outgoing wave fields, respectively, where the terms "incoming" and "outgoing" refer now to the exterior domain $\mathbb{R}^n\setminus\Omega$ rather than interior domain Ω as before.

It is important to note that when constructing both $\boldsymbol{P}_\Omega$ and $\boldsymbol{P}_\Gamma$ (see section 3.2) and $\boldsymbol{Q}_{\mathbb{R}^n\setminus\Omega}$ and $\boldsymbol{Q}_\Gamma$ (see above), we use the same operator $\boldsymbol{Tr}$ of (3.4), i.e., the same direction $\boldsymbol{n}$ of differentiation at the boundary Γ. This basically creates an asymmetry because for one domain the normal is directed inwards and for the other one, outwards. However, this asymmetry allows us to always consider the same space of boundary functions Ξ_Γ and thus arrive at convenient relations: $\boldsymbol{P}_\Gamma = \boldsymbol{I} - \boldsymbol{Q}_\Gamma$, and decompositions (3.13) and (3.17). Hereafter, we will always assume that the direction of normal differentiation at the boundary Γ is the same for both the interior and exterior domains.

Representation (3.17) suggests that one can also describe the controls $g(x)$ for the interior domain Ω in terms of the potential $\boldsymbol{Q}_{\mathbb{R}^n\setminus\Omega}$ and projection $\boldsymbol{Q}_\Gamma$ constructed for the exterior domain $\mathbb{R}^n\setminus\Omega$. Indeed, $\operatorname{Ker}\boldsymbol{Q}_\Gamma$ contains those and only those boundary functions that can be interpreted as traces of the outgoing waves with respect to the domain $\mathbb{R}^n\setminus\Omega$. The sole task of the controls $g(x)$ is to eliminate the influence that domain $\mathbb{R}^n\setminus\Omega$ exerts on the domain Ω or, in other words, *eliminate the component of*

the acoustic field that is outgoing with respect to $\mathbb{R}^n \backslash \Omega$. Thus, given the total field $\xi_\Gamma = (u, \frac{\partial u}{\partial n})|_\Gamma$ on the boundary, the control has to respond to its outgoing component $(\boldsymbol{I} - \boldsymbol{Q}_\Gamma)\xi_\Gamma \in \operatorname{Ker} \boldsymbol{Q}_\Gamma$ (with respect to the domain $\mathbb{R}^n \backslash \Omega$) and actually eliminate it. In other words, it has to reconstruct $-(\boldsymbol{I} - \boldsymbol{Q}_\Gamma)\xi_\Gamma$ from the given data ξ_Γ; the application of the operator $-(\boldsymbol{I} - \boldsymbol{Q}_\Gamma)$ via auxiliary function $w(x)$ immediately yields the set of controls in the form (2.20).

We finally note that the construction of $\boldsymbol{Q}_{\mathbb{R}^n \backslash \Omega}$ and $\boldsymbol{Q}_\Gamma$, which is fully parallel to that of $\boldsymbol{P}_\Omega$ and $\boldsymbol{P}_\Gamma$, essentially lifts the assumption made in the beginning of section 1.2 that the domain to be shielded should be bounded.

In the following section, we summarize in general terms the motivation for and benefits of using Calderon's potentials for analyzing noise control problems.

3.4. Calderon's equations versus integral equations of classical potential theory. One obvious reason for introducing Calderon's potentials in the framework of active noise control is compactness and convenience of all representations. This has already been illustrated by the previous discussion and will be further stressed by the material of section 4. In addition to that, there are two fundamental advantages of using Calderon's potentials.

First, let us emphasize that in this paper we do not solve boundary-value problems per se. All the problems that we study here are formulated on the entire $\mathbb{R}^n$ ($n = 2$ or $n = 3$) and driven by a particular collection of sources introduced as right-hand sides to the governing differential equation: friendly sources, adverse sources, and control sources. Unique solvability of such problems in the simplest case of the Helmholtz equation, as in sections 2 and 3, is guaranteed by the Sommerfeld conditions (2.2) that manifest the so-called principle of limiting absorption; see, e.g., [24]. In the more complex cases of section 4 (see below), we make the assumption of unique solvability ahead of time, which is natural from the standpoint of physics.

On the other hand, even so, we do not solve boundary-value problems; the control sources $g(x)$ are generated based on a particular set of quantities defined (in practice, measured) on the boundary Γ. These are the solution u itself and its normal derivative (see (2.14), (2.20)) and *it is very important that both quantities be taken into account.* Otherwise, for the Helmholtz equation we could have encountered the problem of resonances. For example, if $-k^2$ (see (1.1)) is an eigenvalue of the Dirichlet problem for the Laplace operator on a bounded domain Ω, then the corresponding Helmholtz problem does not have a unique solution. This phenomenon is called internal resonance. In this case, we obviously cannot use classical potentials and parameterize the entire variety of Helmholtz solutions on Ω by scalar Dirichlet data on Γ only. As such, these boundary data will simply be insufficient to control Helmholtz solutions on Ω (either directly or through building volumetric controls based on these surface data). Similarly, if $-k^2$ were an eigenvalue of the Neumann problem for the Laplace operator, then a set of scalar Neumann data only (i.e., normal derivative on Γ) would be insufficient to control Helmholtz solutions on Ω. In some other, nonresonant cases, it may basically be possible to develop controls based on one scalar boundary quantity only. In general, however, to derive controls "from the surface to the domain," we would like to have a universal equivalent boundary parameterization of the entire variety of Helmholtz solutions on Ω. This is provided by Calderon's BEP (3.6) and potentials (3.3) that always have *vector boundary density* ξ_Γ. In this framework, there may not be a nonzero solution on Ω with zero boundary trace ξ_Γ. For the aforementioned example with resonant interior Dirichlet problem this means that even if the adverse component of the acoustic field (i.e., noise) to be canceled on Ω turns into zero on

Γ, this noise is still going to be sensed by our methodology through its nonzero normal derivative, and then the annihilating control sources can be derived accordingly. Therefore, we can say that by providing an equivalent boundary parameterization of all solutions on the domain (see Proposition 3.1) and, as a consequence, a universal classification of *all waves* into incoming and outgoing (see (3.13) and (3.17)), Calderon's potentials help us avoid problems related to resonances.

Of course, one can say that boundary representations that use vector traces and, as such, guarantee for the Helmholtz equation the equivalence stated in Proposition 3.1, have, in fact, already been obtained in section 2 with the help of classical Green's formulae. In this sense, the formulation of sections 3.2 and 3.3 that introduces the split between incoming and outgoing waves using Calderon's projections can simply be regarded as an alternative, more convenient way to express the same ideas as in section 2. However, Calderon's potentials (in the form of [28, 29, 30]) will allow us to encompass a much wider class of formulations than classical potential theory can handle. In so doing, we will be able to employ the same key concept of split between incoming and outgoing waves, which in many cases may be all but straightforward. This is the second major advantage of applying Calderon's theory to noise control problems.

Indeed, the equations of classical potential theory require explicit knowledge of the fundamental solution, e.g., (2.1). This automatically limits the consideration to the case of constant-coefficient differential equations with convenient-to-analyze far-field boundary conditions, e.g., the Sommerfeld conditions (2.2) at infinity. In the following section, we will see that when using Calderon's potentials one basically does not need the fundamental solution. Moreover, instead of boundary conditions (2.2) only, Calderon's theory will allow us to consider general well-posed types of far-field boundary conditions that are set not necessarily at infinity but maybe at a remote far-field boundary. Finally, Calderon's potentials will also allow us to handle equations with variable (including discontinuous) coefficients. Clearly, neither of the aforementioned cases can be treated in the framework of classical potential theory.

There is another reason for introducing Calderon's potentials, which is related more to future work rather than to specific topics considered in this paper. Namely, hereinafter we analyze time-harmonic acoustic fields, i.e., essentially study the standing waves, while temporal behavior of the solution is characterized by one constant frequency, which is assumed to be known. A physically more interesting and mathematically more challenging problem is to develop active controls in the genuinely unsteady case, i.e., for the broad-band spectra of frequencies. Even though so little work has been done as of yet along these specific lines, there is strong indication that general constructions of Calderon's potentials can be rather naturally carried over to the unsteady formulation; see, e.g., [32, 33]. These time-dependent Calderon's equations will, in particular, account for fundamental issues of causality. Consequently, we expect that in the future similar analysis tools will allow us to study time-dependent noise control problems as well.

Altogether, we conclude that Calderon's potentials and boundary projections provide a convenient and powerful apparatus for analyzing a series of consecutively more difficult noise control problems in the same uniform manner.

4. A more general formulation of the active shielding problem. Here we discuss several important generalizations to the formulation of the noise control problem that we have studied in sections 2 and 3.

4.1. Changing the far-field boundary conditions. Instead of the entire space $\mathbb{R}^n$, consider a sufficiently large domain Ω_0, $\Omega \subset \Omega_0 \subseteq \mathbb{R}^n$. We introduce a setup similar to that of section 1.2. Namely, the total acoustic field $u(x)$ is governed by the Helmholtz equation (1.1) and the sources $f(x)$ that drive the solution are again split into the interior and exterior components (see (1.2)): $\operatorname{supp} f^+ \subset \Omega$ as before, but as opposed to section 1.2, $\operatorname{supp} f^- \subset \Omega_0 \backslash \Omega$ (rather than $\mathbb{R}^n \backslash \Omega$). Also different is the far-field boundary condition. Previously we required that the solution $u(x)$ satisfy the Sommerfeld condition (2.2a) or (2.2b); now we specify a general linear homogeneous boundary condition at $\partial \Omega_0$, which we formulate as an inclusion,

$$(4.1) \qquad\qquad u \in U_0,$$

and require that (1.1) subject to boundary condition (4.1) be uniquely solvable on the domain Ω_0 for any (sufficiently smooth) right-hand side $f(x)$. Similar to section 1.2, we decompose the overall solution into the friendly component $u^+(x)$, $x \in \Omega_0$, which is called sound, and adverse component $u^-(x)$, $x \in \Omega_0$, which is called noise; see formulae (1.3), (1.4). Again similar to section 1.2, we would like to construct a control function $g(x)$, $\operatorname{supp} g \subset \Omega_0 \backslash \Omega$, so that the total acoustic field $\tilde{u}(x)$, which then includes the output of controls as well and is governed by (1.5), will coincide on the domain Ω with only the sound component $u^+(x)$, i.e., satisfy equality (1.6). We denote by $\boldsymbol{G}$ the inverse operator to $\boldsymbol{L}$ so that $u = \boldsymbol{G} f$, $u \in U_0$; the aforementioned unique solvability means, in particular, that for any $w \in U_0$ we have

$$(4.2) \qquad\qquad w = \boldsymbol{G}\boldsymbol{L}w.$$

In the previously analyzed case, the space U_0 (see (4.1)) contained all those and only those functions that were defined on $\mathbb{R}^n$ and satisfied the corresponding Sommerfeld condition (2.2a) or (2.2b), and the inverse operator $\boldsymbol{G}$ was constructed as a convolution with the fundamental solution (2.1a) or (2.1b). As for the uniqueness in the form (2.17), it has now transformed into (4.2).

Using the inverse operator $\boldsymbol{G}$, we will now formally construct the potentials and projections for both domains, Ω and $\Omega_0 \backslash \Omega$, similarly to the way it was done in section 3. Given a boundary density $\xi_\Gamma = (\xi_0, \xi_1)$, we take some sufficiently smooth auxiliary function $w \in U_0$ that satisfies condition (3.7) and define

$$\boldsymbol{P}_\Omega \xi_\Gamma(x) = w - \boldsymbol{G}\left\{\boldsymbol{L}w\big|_\Omega\right\}$$

$$(4.3\text{a})$$

$$= \boldsymbol{G}\left\{\boldsymbol{L}w\big|_{\Omega_0 \backslash \Omega}\right\}, \qquad x \in \Omega,$$

$$(4.3\text{b}) \qquad\qquad \boldsymbol{P}_\Gamma \xi_\Gamma = \boldsymbol{Tr}\,\boldsymbol{P}_\Omega \xi_\Gamma.$$

Definitions (4.3a) and (4.3b) are obviously analogous to (3.8) and (3.5), respectively. Expressions in curly brackets in (4.3a) mean that after applying the differential operator $\boldsymbol{L}$ to the function $w(x)$ the result is truncated to the corresponding domain, either Ω or $\Omega_0 \backslash \Omega$. The second equality in (4.3a) is guaranteed by the uniqueness (4.2).

Similarly, for the complementary domain $\Omega_0 \backslash \Omega$ we have

$$\boldsymbol{Q}_{\Omega_0 \backslash \Omega} \xi_\Gamma(x) = w - \boldsymbol{G}\left\{\boldsymbol{L}w\big|_{\Omega_0 \backslash \Omega}\right\}$$

$$(4.4\text{a})$$

$$= \boldsymbol{G}\left\{\boldsymbol{L}w\big|_\Omega\right\}, \qquad x \in \Omega_0 \backslash \Omega,$$

$$(4.4\text{b}) \qquad\qquad \boldsymbol{Q}_\Gamma \xi_\Gamma = \boldsymbol{Tr}\,\boldsymbol{Q}_{\Omega_0 \backslash \Omega} \xi_\Gamma.$$

Again, the definitions (4.4a) and (4.4b) are analogous to (3.15) and (3.16), respectively, and the second equality in (4.4a) follows from (4.2).

The definitions (4.3) and (4.4) are so far only formal. To make them meaningful, we need to ensure that the projections (4.3b) and (4.4b) indeed split the overall wave field into the incoming and outgoing components, as before. In other words, we need to guarantee the key result similar to that of Proposition 3.1. This, in fact, requires that the potentials (4.3a) and (4.3b) depend only on the density ξ_Γ and be independent of the particular choice of $w(x)$ as long as it satisfies (3.7). In other words, we need to require that if $\xi_\Gamma = 0$, then for any function $w(x)$, $w \in U_0$, with the zero trace $\boldsymbol{Tr}\, w = \xi_\Gamma = 0$ (see (3.4)), we would obtain

$$(4.5a) \qquad \boldsymbol{P}_\Omega \xi_\Gamma = \boldsymbol{P}_\Omega 0_\Gamma = w - \boldsymbol{G} \left\{ \boldsymbol{L}\, w \big|_\Omega \right\} = 0, \qquad x \in \Omega,$$

and

$$(4.5b) \qquad \boldsymbol{Q}_{\Omega_0 \backslash \Omega} \xi_\Gamma = \boldsymbol{Q}_{\Omega_0 \backslash \Omega} 0_\Gamma = w - \boldsymbol{G} \left\{ \boldsymbol{L}\, w \big|_{\Omega_0 \backslash \Omega} \right\} = 0, \quad x \in \Omega_0 \backslash \Omega.$$

One can show that for the Helmholtz operator $\boldsymbol{L}$ of (1.1) and boundary condition (4.1) that guarantees the unique solvability, equalities (4.5) will always hold. Alternatively, we can say that a particular choice of the trace according to (3.4) is proper for the given differential operator $\boldsymbol{L}$ in the sense that zero trace implies zero potential. The same is, in fact, true for many other linear elliptic differential operators of the second order. For details, we refer the reader to [29], where requirements of the type (4.5) are used for characterizing a special class of boundary traces, the so-called *clear traces*, which are then employed for building the generalized potentials and projections of Calderon's type. In the current paper we restrict ourselves to considering only the traces of type (3.4) (Cauchy data), which are applicable, in particular, to the second-order time-harmonic wave problems that we are solving.

Having constructed the potentials and projections (4.3) and (4.4) based on the operator $\boldsymbol{G}$ that corresponds to the general boundary condition (4.1), we can now develop the full set of arguments similar to that of section 3 provided that equalities (4.5) hold. In particular, the annihilating function $v(x)$ for the interior domain Ω is still given by formula (3.10) with the only difference that the potential $\boldsymbol{P}_\Omega$ is now defined by (4.3a). There is still a split (3.13) of the total acoustic field into the incoming and outgoing components with respect to the domain Ω; however, the split is now rendered by the projection $\boldsymbol{P}_\Gamma$ defined by (4.3b) via the operator $\boldsymbol{G}$ and thus differs from that of section 3. Similarly, there is a split (3.17) of the total acoustic field into the incoming and outgoing components with respect to the domain $\Omega_0 \backslash \Omega$, although the projection $\boldsymbol{Q}_\Gamma$ defined by (4.4b) via the operator $\boldsymbol{G}$ is again different from that of section 3.

Since the projection operators $\boldsymbol{P}_\Gamma$ and $\boldsymbol{Q}_\Gamma = \boldsymbol{I} - \boldsymbol{P}_\Gamma$ are now not the same as those of section 3, so are the results $\boldsymbol{P}_\Gamma \xi_\Gamma$ and $\boldsymbol{Q}_\Gamma \xi_\Gamma$ of applying these operators to a given ξ_Γ. In other words, the incoming and outgoing parts of a given ξ_Γ obtained by applying the operator $\boldsymbol{P}_\Gamma$ of (4.3b) (or $\boldsymbol{Q}_\Gamma$ of (4.4b)) *will not, generally speaking, be the same* as the incoming and outgoing parts, respectively, obtained in section 3. This means that the decomposition of the total acoustic field into the incoming and outgoing components *depends on the external boundary condition* (4.1) that these components satisfy, which is, of course, reasonable from the standpoint of physics. However, one can show that although the change of the domain and external far-field boundary condition introduced here compared to section 3 does amount to the

change of the projection operators $\boldsymbol{P}_\Gamma$ and $\boldsymbol{Q}_\Gamma$, *it does not induce any changes of the subspace* $\operatorname{Im} \boldsymbol{P}_\Gamma = \operatorname{Ker} \boldsymbol{Q}_\Gamma$. As concerns the other subspace of the split (3.13) or (3.17), $\operatorname{Ker} \boldsymbol{P}_\Gamma = \operatorname{Im} \boldsymbol{Q}_\Gamma$, it does, generally speaking, change. This can be interpreted as follows. $\operatorname{Im} \boldsymbol{P}_\Gamma$ contains all those and only those boundary functions from Ξ_Γ that can be complemented on the interior domain Ω to the solution of the homogeneous equation $\boldsymbol{L}u = 0$. This set of functions as a whole will not, of course, depend on any external boundary condition. However, we actually construct this set by applying the projection $\boldsymbol{P}_\Gamma$ to the entire space Ξ_Γ. The direction of projecting depends on the external boundary conditions. This dependency manifests itself through the change in the operators, and for a given $\xi_\Gamma \in \Xi_\Gamma$ we will, generally speaking, obtain a different $\boldsymbol{P}_\Gamma \xi_\Gamma$, which will still belong, however, to the same subspace $\operatorname{Im} \boldsymbol{P}_\Gamma$ that does not change. The change in the direction of projecting onto this subspace, $\operatorname{Im} \boldsymbol{P}_\Gamma$, obviously implies the change in the second component of the direct sum (3.13) (or (3.17)). This second component $\operatorname{Ker} \boldsymbol{P}_\Gamma = \operatorname{Im} \boldsymbol{Q}_\Gamma$ contains all those and only those functions from Ξ_Γ that can be complemented on $\Omega_0 \backslash \Omega$ to a solution of $\boldsymbol{L}u = 0$ that satisfies (4.1); these functions will, of course, depend on the actual exterior domain and far-field boundary condition.

Let us now emphasize the most important thing regarding the relation between the potential and projection operators that we build according to formulae (4.3) and (4.4) (cf. (3.2), (3.5) and (3.15), (3.16)) and the noise control problem that we analyze using these operators. It turns out that *although the potentials and projections change with the change of the external boundary condition and so does the decomposition of the acoustic field into the incoming and outgoing components (with respect to Ω or $\Omega_0 \backslash \Omega$), the set of control functions $\{g(x)\}$ that we obtain is essentially insensitive to these changes.* Indeed, as the annihilating function $v(x)$ is given by (3.10) with the new potential defined by (4.3a), then similarly to (2.20) and Proposition 2.1 the entire set of appropriate controls for Ω is given by

$$(4.6) \qquad g(x) = -\boldsymbol{L}w\Big|_{x \in \Omega_0 \backslash \Omega},$$

where $w(x)$ satisfies boundary condition (4.1) and also (2.14) (the latter can be rewritten as (3.14)). Thus, the only difference between (2.20) and (4.6) is that in (2.20) $w(x)$ is supposed to satisfy one of the Sommerfeld conditions (2.2a) or (2.2b) rather than condition (4.1). On the other hand, for applications we will primarily be interested in controls concentrated near the perimeter of the region to be shielded, i.e., near the boundary $\Gamma = \partial\Omega$, rather than far away from it. Therefore, we can always consider only those $w(x)$ that are compactly supported near Γ. Such functions will obviously satisfy any homogeneous far-field boundary condition and thus the set of controls built on the basis of these compactly supported $w(x)$'s will be exactly the same if we use either of the formulae (2.20) or (4.6).

4.2. Changing the differential operator. Instead of the original Helmholtz operator (see (1.1)), we will now consider some other linear differential operator that operates on the functions defined on the domain Ω_0. We will keep the same notation $\boldsymbol{L}$ for simplicity and require that the acoustic field $u(x)$ governed by the new equation $\boldsymbol{L}u = f$ satisfy the same far-field boundary condition (4.1) as before. Similar to section 4.1, we will require that the equation $\boldsymbol{L}u = f$ subject to boundary condition (4.1) be uniquely solvable for any sufficiently smooth right-hand side f, and also for simplicity, we will keep the same notation $\boldsymbol{G}$ for the corresponding inverse operator. From the standpoint of the physical model, the change of the operator may

be accounted for, e.g., by the change in the properties of the medium across which the acoustic signals propagate. For example, as opposed to the previously analyzed case of the classical constant-coefficient Helmholtz equation that describes stationary waves in an isotropic medium, we may consider anisotropies of the medium, e.g., obstacles like passenger seats and overhead bins that can be introduced in the passenger compartment of an aircraft. Basically, obstacles, i.e., anisotropies, of various nature can be introduced in either of the domains Ω or $\Omega_0 \backslash \Omega$ or in both of them. At any rate, provided that the regime of the wave propagation is still linear (some questions related to nonlinearities are touched upon in section 4.4), we arrive at a new equation that may have variable coefficients in the areas where the obstacles are introduced. In this section, we limit our analysis to the case in which these coefficients are still smooth across the boundary Γ; the case in which discontinuities in the material properties are allowed along Γ is considered in section 4.3.

Using the new operators $\boldsymbol{L}$ and $\boldsymbol{G}$ we can repeat the entire analysis of section 4.1. In so doing, we arrive at the operators $\boldsymbol{P}_\Omega$, $\boldsymbol{P}_\Gamma$ and $\boldsymbol{Q}_{\Omega_0 \backslash \Omega}$, $\boldsymbol{Q}_\Gamma$ defined by the same formulae (4.3) and (4.4), respectively, but based on the new $\boldsymbol{L}$ and $\boldsymbol{G}$. Provided that relations (4.5) hold, we can guarantee the results similar to those of Proposition 3.1 and thus decompose the overall acoustic field into the incoming and outgoing components. (The terms "incoming" and "outgoing" in this section still relate to a particular domain, Ω or $\Omega_0 \backslash \Omega$, but the field itself and thus each of its components is governed by a different equation.) As has been mentioned (see section 4.1 and also [29] for details), relations (4.5) essentially imply that the trace operator $\boldsymbol{Tr}$ is correlated in some sense with the differential operator $\boldsymbol{L}$. In this paper we are using only the traces of type (3.4), which are consistent (see [29]), in particular, with the second-order linear elliptic differential operators $\boldsymbol{L}$, for which the Neumann boundary data reduce to a standard normal derivative, i.e., $\boldsymbol{L}u = \nabla (p \nabla u) + \{\text{lower order terms}\}$, $p = p(x)$ is a scalar function. Thus, the changes of the operator that we are considering shall only be within this class. This is obviously going to be sufficient for all practical purposes related to time-harmonic acoustics.

Of course, the potentials and projections built on the basis of the new operators $\boldsymbol{L}$ and $\boldsymbol{G}$ will not be the same as those of either section 4.1 or section 3. Similar to the considerations of section 4.1, one can see that changing $\boldsymbol{L}$ only in the exterior domain $\Omega_0 \backslash \Omega$ will change both $\boldsymbol{P}_\Gamma$ and $\boldsymbol{Q}_\Gamma$, but $\text{Im} \boldsymbol{P}_\Gamma = \text{Ker} \boldsymbol{Q}_\Gamma$ will remain the same, whereas $\text{Ker} \boldsymbol{P}_\Gamma = \text{Im} \boldsymbol{Q}_\Gamma$ will, generally speaking, change. (The change of $\text{Ker} \boldsymbol{P}_\Gamma = \text{Im} \boldsymbol{Q}_\Gamma$ accounts, as before, for a different direction of projecting onto $\text{Im} \boldsymbol{P}_\Gamma$.) Changing $\boldsymbol{L}$ only on the interior domain Ω will again cause changes to both $\boldsymbol{P}_\Gamma$ and $\boldsymbol{Q}_\Gamma$, but $\text{Im} \boldsymbol{Q}_\Gamma = \text{Ker} \boldsymbol{P}_\Gamma$ will not change (whereas $\text{Ker} \boldsymbol{Q}_\Gamma = \text{Im} \boldsymbol{P}_\Gamma$ will). *However, neither of these changes will essentially affect the control functions $g(x)$.*

Indeed, to cancel out the unwanted noise on the interior domain Ω, we use the annihilating function $v(x) = -\boldsymbol{P}_\Omega \xi_\Gamma$ with $\xi_\Gamma = (u, \frac{\partial u}{\partial \boldsymbol{n}})|_\Gamma$ (see (3.10)) and, accordingly, arrive at the control functions $g(x)$ given by formula (4.6), where $\boldsymbol{L}$ is now the new operator and $w(x)$ is a sufficiently smooth function that satisfies (4.1) and (2.14). Formula (4.6) still gives the general solution for controls and formally looks the same as before with the "hidden" difference in plugging in the new $\boldsymbol{L}$. The reason that the controls are described by similar formulae even when the actual wave propagation processes are governed by different equations is that *in every particular instance both the unwanted disturbances to be canceled and the output of controls propagate across the same medium and satisfy the same far-field boundary conditions.* It turns out that having realized it we can design the controls *with no explicit knowledge of the actual*

properties of the supporting medium, i.e., the coefficients of $\boldsymbol{L}$, except in the region where $w(x) \neq 0$, i.e., where we apply the operator $\boldsymbol{L}$; see (4.6).

As has been mentioned, the region where $w(x) \neq 0$ may always be chosen as a narrow strip straddling the boundary Γ. (This is beneficial from the standpoint of practical design.) Therefore, in the particular case when the aforementioned obstacles (either in Ω or in $\Omega_0 \backslash \Omega$ or in both domains) are introduced not right next to the boundary but rather at a distance, the operator $\boldsymbol{L}$ near the boundary remains the same Helmholtz operator of (1.1), and thus the exact same control functions as those obtained in sections 2 and 3 on the basis of compactly supported $w(x)$'s can be used for noise cancellation in the current more general setting.

4.3. Introducing discontinuities in the material properties. The next generalization that comes naturally after considering the changes of the far-field boundary conditions (section 4.1) and the changes of the coefficients of $\boldsymbol{L}$ that may be caused by anisotropies in Ω and/or $\Omega_0 \backslash \Omega$ (section 4.2) is to consider two different media separated by the boundary Γ. In some sense, this is the ultimate change of the operator that we may consider; it further extends the analysis of section 4.2, in which we allowed for the variation of the coefficients of $\boldsymbol{L}$ but still assumed that they were smooth across the boundary Γ. In this section, we allow the coefficients of the operator $\boldsymbol{L}$ to have jumps on the interface Γ (e.g., the media on different sides of Γ may have different refraction indices k; see (1.1)). Thus, some additional interface conditions may, generally speaking, be required for solvability. In many cases, and in particular those that we are studying in this paper, these additional conditions are the continuity of the solution itself and the continuity of its normal flux across the interface. For the second-order operators, we can write it in the general form as follows:

$$(4.7) \qquad\qquad \xi_\Gamma^{(\Omega_0 \backslash \Omega)} = \boldsymbol{B} \xi_\Gamma^{(\Omega)},$$

where $\xi_\Gamma^{(\Omega)} = \boldsymbol{Tr}^{(\Omega)} u$ and $\xi_\Gamma^{(\Omega_0 \backslash \Omega)} = \boldsymbol{Tr}^{(\Omega_0 \backslash \Omega)} u$ are traces of the solution $u(x)$ in the sense of (3.4) taken from the interior and exterior sides of Γ, respectively, and $\boldsymbol{B}$ is a linear operator that provides the required relation between these traces (e.g., continuity of the solution and its normal flux on Γ).

We require that the differential equation $\boldsymbol{L}u = f$ be uniquely solvable on the domain Ω_0 for any sufficiently smooth right-hand side f in the class of functions that satisfy the far-field boundary condition (4.1) and the interface compatibility condition (4.7). We denote by $\boldsymbol{G}$ the corresponding inverse operator to $\boldsymbol{L}$ so that $u = \boldsymbol{G}f$, $u \in U_0$, and u satisfies (4.7). The aforementioned unique solvability implies, in particular, that relation (4.2) will hold for any function $w(x)$ that satisfies (4.1) and (4.7).

To build the Calderon's operators, we now need to distinguish between the boundary traces on different sides of Γ. Namely, given the boundary density $\xi_\Gamma^{(\Omega)}$, we take an arbitrary (sufficiently smooth) auxiliary function $w(x) \in U_0$, $\boldsymbol{Tr}^{(\Omega)} w = \xi_\Gamma^{(\Omega)}$, so that it also meets the interface condition (4.7) (in the sense that the two functions $\xi_\Gamma^{(\Omega_0 \backslash \Omega)} = \boldsymbol{Tr}^{(\Omega_0 \backslash \Omega)} w$ and $\xi_\Gamma^{(\Omega)}$ are connected via (4.7)) and construct the operators $\boldsymbol{P}_\Omega$ and $\boldsymbol{P}_\Gamma$ according to formulae (4.3); the boundary trace in formula (4.3b) is taken from the side of the domain Ω. Again, the condition of (4.5a) type is required to guarantee the result of Proposition 3.1. Similarly, we obtain $\boldsymbol{Q}_{\Omega_0 \backslash \Omega}$ and $\boldsymbol{Q}_\Gamma$ according to formulae (4.4); these operators operate on $\xi_\Gamma^{(\Omega_0 \backslash \Omega)}$ and the auxiliary function $w(x) \in U_0$, $\boldsymbol{Tr}^{(\Omega_0 \backslash \Omega)} w = \xi_\Gamma^{(\Omega_0 \backslash \Omega)}$, needed for the calculation is again taken so that it

meets the interface condition (4.7). Proposition 3.1 will hold provided equality (4.5b) is satisfied.

One important difference compared to the previously analyzed cases is that here, generally speaking, $\boldsymbol{I} - \boldsymbol{P}_\Gamma \neq \boldsymbol{Q}_\Gamma$ and therefore, *we can no longer claim that* $\mathrm{Im}\boldsymbol{P}_\Gamma = \mathrm{Ker}\boldsymbol{Q}_\Gamma$ *and* $\mathrm{Ker}\boldsymbol{P}_\Gamma = \mathrm{Im}\boldsymbol{Q}_\Gamma$. In fact, similar to what we had before, all those and only those boundary functions $\xi_\Gamma^{(\Omega)}$ that satisfy the BEP $\boldsymbol{P}_\Gamma \xi_\Gamma^{(\Omega)} = \xi_\Gamma^{(\Omega)}$, i.e., $\xi_\Gamma^{(\Omega)} \in \mathrm{Im}\boldsymbol{P}_\Gamma$, can be complemented on Ω to the solution of the homogeneous equation $\boldsymbol{L}u = 0$ (Proposition 3.1); these functions $\xi_\Gamma^{(\Omega)} \in \mathrm{Im}\boldsymbol{P}_\Gamma$ should still be interpreted as traces of the incoming waves with respect to the domain Ω. Moreover, the projection operator $\boldsymbol{P}_\Gamma$ still renders the split of the overall acoustic field into the direct sum of its incoming and outgoing components with respect to the domain Ω. However, contrary to what we had before, we now need to treat this overall acoustic field $\xi_\Gamma^{(\Omega)}$ as defined, i.e., "measured," on the interface Γ from its interior side and thus rewrite formula (3.13) as $\Xi_\Gamma^{(\Omega)} = \mathrm{Im}\boldsymbol{P}_\Gamma \oplus \mathrm{Ker}\boldsymbol{P}_\Gamma$. Moreover, the functions $\xi_\Gamma^{(\Omega)} \in \mathrm{Ker}\boldsymbol{P}_\Gamma$ that are interpreted as traces of the outgoing waves with respect to Ω, cannot, generally speaking, be complemented to the solution of the homogeneous equation $\boldsymbol{L}u = 0$ on $\Omega_0 \backslash \Omega$, i.e., do not belong to $\mathrm{Im}\boldsymbol{Q}_\Gamma$. This "mismatch" between $\mathrm{Ker}\boldsymbol{P}_\Gamma$ and $\mathrm{Im}\boldsymbol{Q}_\Gamma$ may give rise to reflections from the interface when analyzing a particular problem.

Of course, the considerations that we have just brought forward are fully reciprocal with respect to the exterior domain $\Omega_0 \backslash \Omega$. In other words, we rewrite the decomposition formula (3.17) as $\Xi_\Gamma^{(\Omega_0 \backslash \Omega)} = \mathrm{Im}\boldsymbol{Q}_\Gamma \oplus \mathrm{Ker}\boldsymbol{Q}_\Gamma$, emphasizing that we now measure the overall solution from the exterior side of the interface Γ, and note that the outgoing waves $\xi_\Gamma^{(\Omega_0 \backslash \Omega)} \in \mathrm{Ker}\boldsymbol{Q}_\Gamma$ may be (partially) reflected from the interface because $\mathrm{Ker}\boldsymbol{Q}_\Gamma \neq \mathrm{Im}\boldsymbol{P}_\Gamma$.

As concerns the set of controls $\{g(x)\}$, however, it still remains essentially the same as before and neither of the aforementioned changes in the formulation of the problem and solution structure, including possible reflections from the interface, actually affect it. Indeed, the annihilating function for the domain Ω is given by $v(x) = -\boldsymbol{P}_\Omega \xi_\Gamma^{(\Omega)}$, where $\xi_\Gamma^{(\Omega)} = (u, \frac{\partial u}{\partial \boldsymbol{n}})|_\Gamma^{(\Omega)}$ (see (3.10)); i.e., we use the generalized Calderon's potential with the density obtained as a trace of the overall acoustic field measured on the interior side of Γ. Using equalities (4.3a), we rewrite it as

$$(4.8) \qquad\qquad v(x) = -\boldsymbol{G}\left\{\boldsymbol{L}w\big|_{\Omega_0 \backslash \Omega}\right\},$$

where the auxiliary function $w(x)$ belongs to U_0 (see (4.1)), $\boldsymbol{Tr}^{(\Omega)}w = \xi_\Gamma^{(\Omega)}$, and also satisfies the interface condition (4.7). Accordingly, the control function $g(x)$ is again given by formula (4.6), where we can always take $w(x)$ compactly supported near Γ. Moreover, from representation (4.8) for the annihilating function we conclude that we do not even need to know the entire $w(x)$; it is sufficient to construct this auxiliary function only on the exterior domain $\Omega_0 \backslash \Omega$, making sure that $\boldsymbol{Tr}^{(\Omega_0 \backslash \Omega)}w = \boldsymbol{B}\xi_\Gamma^{(\Omega)} = \xi_\Gamma^{(\Omega_0 \backslash \Omega)}$, i.e., that it satisfies the correct interface compatibility condition (4.7). In practice, the trace of the solution at the boundary represents the values that are actually measured. *Therefore, we are building the controls according to formula* (4.6), *where the auxiliary function* $w(x)$ *is "tuned" in the sense of* $\boldsymbol{Tr}^{(\Omega_0 \backslash \Omega)}w = \xi_\Gamma^{(\Omega_0 \backslash \Omega)}$ *to the trace of the actual acoustic field measured on the exterior side of* Γ.

Moreover, representation (4.8) for the annihilating function $v(x)$ along with formulae (4.4) and (3.17) suggest that this annihilating function is, in fact, constructed to eliminate the outgoing waves with respect to the domain $\Omega_0 \backslash \Omega$. This interpretation

has already been mentioned at the end of section 3.3; however, we then analyzed the case in which the outgoing waves with respect to $\Omega_0 \backslash \Omega$ were the same as the incoming waves with respect to Ω. In the current framework this is no longer so, and the waves generated by the sources f^- not only enter the domain Ω but may also be reflected back from the interface Γ. The controls that we build according to formula (4.6) respond to the actual field measured on the exterior side of Γ, $\boldsymbol{Tr}^{(\Omega_0 \backslash \Omega)} w = \xi_\Gamma^{(\Omega_0 \backslash \Omega)}$, *and cancel out the entire outgoing field component with respect to $\Omega_0 \backslash \Omega$; i.e., both the portion of the field that propagates through the boundary Γ into Ω and the portion that may be reflected back to $\Omega_0 \backslash \Omega$.* Note that the reflections themselves do not "contaminate" the measurements because they already belong to $\operatorname{Im} \boldsymbol{Q}_\Gamma$ and thus do not affect the output of the controls. We conclude that in some sense it may be more natural to describe the controls in terms of the outgoing waves with respect to $\Omega_0 \backslash \Omega$ rather than incoming waves with respect to Ω, especially in the case when these waves are not the same.

4.4. Introducing nonlinearities. A recipe for including nonlinearities in the formulation of the active shielding problem that we analyzed in this paper was first proposed by Ryaben'kii in [22]. In this section, we simply mention with no detail that the last argument of section 4.3 actually suggests that nonlinearities of a particular nature can be handled by the same controls. Indeed, the controls are designed to cancel out the entire outgoing component of the acoustic field with respect to a particular domain. Clearly, as long as the nonlinearities are not concentrated on or immediately near the perimeter of the region to be shielded, the controls (4.6) will still be responding to the actual acoustic field measured at the boundary and the nature of the sources and the supporting medium away from the boundary—whether they are linear or nonlinear—will not matter. This conclusion holds provided that the nonlinearities satisfy one important limitation, namely, that they do not alter the frequency of the acoustic signals. It is known, however, that many nonlinearities do produce multiple frequencies from a single-frequency input. In this case, provided that the spectrum of frequencies is available ahead of time, the controls can still be built as described above, but for each frequency separately and independently.

5. An example. The synthesis of our control input will be demonstrated on a simple two-dimensional example, where the circular region Ω defined by $r < R$ is protected from exterior noise by active control acting either along the perimeter or within an annular region surrounding Ω. We derive the exact analytic solution in Fourier representation and give explicit expressions for the optimal distributed control. No information about the nature of the exterior sound sources is required, since the solution depends only on the sound field and its normal derivative measured along the perimeter. In actual implementation, this exact control function would be approximated by a finite number of acoustic inputs. The same approach applies to complicated geometries, which can be handled numerically.

5.1. Helmholtz equation in $\mathbb{R}^2$. The Helmholtz equation (1.1) for $n = 2$ expressed in polar coordinates (r, θ) reads

$$(5.1) \qquad u_{rr} + u_r/r + u_{\theta\theta}/r^2 + k^2 u = f,$$

where u is the amplitude of the time-harmonic acoustic field $u\, e^{i\omega t}$, and $k = \omega/c$ is the wavenumber (refraction index).

5.2. Fourier transform. Let $\hat{u}$ be the Fourier transform

$$\hat{u} = \frac{1}{2\pi} \int_0^{2\pi} u \, e^{-im\theta} \, d\theta$$

and

$$\hat{f} = \frac{1}{2\pi} \int_0^{2\pi} f \, e^{-im\theta} \, d\theta.$$

The Helmholtz equation (5.1) in this Fourier representation becomes

$$\text{(5.2)} \qquad\qquad \hat{u}_{rr} + \hat{u}_r/r + (k^2 - m^2/r^2)\hat{u} = \hat{f}.$$

The functions $\hat{u}$ and $\hat{f}$ are associated with a particular mode number m. This is implicit in our notation so that the equations are not cluttered with an excessive number of superscripts.

5.3. Boundary conditions. Functions $\hat{u}$ must be analytic at the origin $r = 0$ and decay to zero at infinity. Moreover, we shall consider compactly supported forcings $\hat{f}$ so that as $r \longrightarrow \infty$, the responses $\hat{u}$ must represent outgoing waves. The outgoing waves satisfy the Sommerfeld condition (2.2a), which means that asymptotically

$$\text{(5.3)} \qquad\qquad \hat{u}(r) \sim \frac{a}{\sqrt{r}} e^{-ikr} \quad \text{as} \quad r \longrightarrow \infty$$

for some constant a.

5.4. Fundamental solution. Let us compute the response to a unit monopole input at $r = s$, $\theta = 0$, i.e., a shifted fundamental solution. (This presents no loss of generality, because for a particular location of the source we can always arrange that its angular coordinate θ be zero by rotating the coordinate system.) We have a δ-type source

$$f(r, \theta) = \delta(r - s)\delta(r\theta) = \delta(r - s)\delta(\theta)/r = \delta(r - s)\delta(\theta)/s$$

so that

$$\text{(5.4)} \qquad\qquad \hat{f}(r) = \frac{\delta(r - s)}{2\pi s} \ .$$

The solution of (5.2) with the right-hand side (5.4) is explicitly constructed as follows. We first notice that (5.2) is homogeneous everywhere except at $r = s$. The homogeneous counterpart to (5.2) has two linearly independent eigensolutions, which we take as $\hat{u}^{(1)}(r) = J_m(kr)$ and $\hat{u}^{(2)}(r) = Y_m(kr) + iJ_m(kr) \equiv iH_m^{(2)}(kr)$, where $J_m(kr)$ and $Y_m(kr)$ are the Bessel functions, so that $\hat{u}^{(1)}(r)$ satisfies the condition of analyticity at $r = 0$ mentioned in section 5.3 and $\hat{u}^{(2)}(r)$ satisfies the condition (5.3). Then, we build the solution of the nonhomogeneous equation (5.2) driven by the δ-source (5.4) from the two branches: $\hat{u}^{(1)}(r)$ for $r < s$ and $\hat{u}^{(2)}(r)$ for $r \geq s$, so that in addition it will be continuous at $r = s$ and have a particular discontinuity in the first derivative prescribed by (5.4). In so doing, we arrive at the fundamental solution

$$\text{(5.5)} \qquad \hat{G}(r, s) = \begin{cases} \dfrac{1}{4} J_m(kr)(Y_m(ks) + iJ_m(ks)) & \text{for } r < s, \\[2mm] \dfrac{1}{4} J_m(ks)(Y_m(kr) + iJ_m(kr)) & \text{for } r \geq s. \end{cases}$$

5.5. Subspaces of incoming and outgoing waves. Let us now consider a distribution of sources on the plane $\mathbb{R}^2$. Then, in the Fourier space we obtain (5.2) driven by the right-hand side $\hat{f}(r)$, which has an extended support on $r \geq 0$, as opposed to the pointwise support as in (5.4). In this case, the corresponding solution is given by the integral ($2\pi s$ is the Jacobian):

$$(5.6) \qquad \hat{u}(r) = \int_{\mathrm{supp}\,\hat{f}} \hat{G}(r,s)\hat{f}(s)\,2\pi s\,ds,$$

which is obviously a "linear combination" of the fundamental solutions $\hat{G}(r,s)$ centered at different locations on $\mathrm{supp}\,\hat{f}$, and which can be considered as a generalization of the standard convolution with the fundamental solution (cf. (2.3)) that we have in the case of constant coefficients.

At a given location R, we can now easily distinguish between the two components of the overall solution $\hat{u}(r)$ given by (5.6): $\hat{u}^+(r)$ generated by the interior sources $\hat{f}^+(s)$, $s < R$, and $\hat{u}^-(r)$ generated by the exterior sources $\hat{f}^-(s)$, $s \geq R$. Indeed, similar to section 2 we have

$$(5.7a) \qquad \hat{u}^+(r) = \int_{\mathrm{supp}\,\hat{f}^+} \hat{G}(r,s)\hat{f}^+(s)\,2\pi s\,ds = \int_{\{s<R\}} \hat{G}(r,s)\hat{f}(s)\,2\pi s\,ds,$$

$$(5.7b) \qquad \hat{u}^-(r) = \int_{\mathrm{supp}\,\hat{f}^-} \hat{G}(r,s)\hat{f}^-(s)\,2\pi s\,ds = \int_{\{s\geq R\}} \hat{G}(r,s)\hat{f}(s)\,2\pi s\,ds.$$

Therefore, we can say that those and only those solutions $\hat{u}^+(r)$ that can be attributed to the interior sources in the sense of representation (5.7a) are "parallel" to the right branch $\hat{u}^{(2)}(r) = Y_m(kr) + iJ_m(kr)$ of the fundamental solution (5.4), i.e., the following Wronskian is equal to zero:

$$(5.8a) \qquad \det \begin{bmatrix} \hat{u}^+ & \hat{u}^{(2)} \\ \dfrac{d\hat{u}^+}{dr} & \dfrac{d\hat{u}^{(2)}}{dr} \end{bmatrix} \Bigg|_{r=R} = 0.$$

Analogously, those and only those solutions $\hat{u}^-(r)$ that can be attributed to the exterior sources (with respect to the location r) in the sense of representation (5.7b) are "parallel" to the left branch $\hat{u}^{(1)}(r) = J_m(kr)$ of the fundamental solution (5.4), i.e., the following Wronskian is equal to zero:

$$(5.8b) \qquad \det \begin{bmatrix} \hat{u}^- & \hat{u}^{(1)} \\ \dfrac{d\hat{u}^-}{dr} & \dfrac{d\hat{u}^{(1)}}{dr} \end{bmatrix} \Bigg|_{r=R} = 0.$$

Clearly, the portion $\hat{u}^+(r)$ of the total solution $\hat{u}(r)$ which is entirely due to the interior sources $s < R$ satisfies the homogeneous counterpart of the differential equation (5.2) on the exterior domain $r \geq R$ and thus, using a standard property of the Wronskian that it is identically zero once it is zero at one point, we obtain from (5.8a)

$$(5.9a) \qquad \frac{d(Y_m(kr) + iJ_m(kr))}{dr}\hat{u}^+ - (Y_m(kr) + iJ_m(kr))\frac{d\hat{u}^+}{dr} = 0, \quad r \geq R.$$

Similarly, the portion $\hat{u}^-(r)$ due to the exterior sources $s \geq R$ alone satisfies

$$(5.9b) \qquad \frac{dJ_m(kr)}{dr}\hat{u}^- - J_m(kr)\frac{d\hat{u}^-}{dr} = 0, \quad r < R.$$

Note that relations of type (5.8) and (5.9) have been used extensively for constructing the so-called artificial boundary conditions that are needed for the numerical solution of infinite-domain problems; see the review paper by Tsynkov [34].

The kernels of the two linear constraints (5.9a) and (5.9b) define a decomposition of the solution space into a direct sum of subspaces corresponding to outgoing and incoming waves, respectively. Note that the concept of splitting the solution into incoming and outgoing components was introduced and explained in detail in section 3.2. Once we identify the location $r = R$ with the boundary of the domain Ω, we can use relations (5.9) to actually decompose the overall acoustic field $\hat{u}(r)$ into the interior and exterior contributions, $\hat{u}^+(r)$ and $\hat{u}^-(r)$, with respect to Ω. This is done by solving (5.9) along with $\hat{u}^+ + \hat{u}^- = \hat{u}$ and $\frac{d\hat{u}^+}{dr} + \frac{d\hat{u}^-}{dr} = \frac{d\hat{u}}{dr}$, i.e., a total of four linear equations, with respect to the unknown quantities $\hat{u}^+$, $\frac{d\hat{u}^+}{dr}$, $\hat{u}^-$, and $\frac{d\hat{u}^-}{dr}$, while treating $\hat{u}$ and $\frac{d\hat{u}}{dr}$ as the given data. The solution is

$$(5.10a) \qquad \hat{u}^+\big|_{r=R} = \frac{\pi r}{2}\left\{\frac{d\hat{u}}{dr}J_m(kr) - \hat{u}\frac{dJ_m(kr)}{dr}\right\}(Y_m(kr) + iJ_m(kr))\Big|_{r=R},$$

$$(5.10b) \qquad \frac{d\hat{u}^+}{dr}\bigg|_{r=R} = \frac{\pi r}{2}\left\{\frac{d\hat{u}}{dr}J_m(kr) - \hat{u}\frac{dJ_m(kr)}{dr}\right\}\frac{d(Y_m(kr) + iJ_m(kr))}{dr}\bigg|_{r=R},$$

$$(5.10c)$$
$$\hat{u}^-\big|_{r=R} = \frac{\pi r}{2}\left\{\hat{u}\frac{d(Y_m(kr) + iJ_m(kr))}{dr} - \frac{d\hat{u}}{dr}(Y_m(kr) + iJ_m(kr))\right\}J_m(kr)\Big|_{r=R},$$

$$(5.10d)$$
$$\frac{d\hat{u}^-}{dr}\bigg|_{r=R} = \frac{\pi r}{2}\left\{\hat{u}\frac{d(Y_m(kr) + iJ_m(kr))}{dr} - \frac{d\hat{u}}{dr}(Y_m(kr) + iJ_m(kr))\right\}\frac{dJ_m(kr)}{dr}\bigg|_{r=R}.$$

5.6. Noise control along the perimeter. In this section, we build the controls concentrated only on the boundary of the domain Ω, i.e., on the circle $r = R$. We consider the situation where the total acoustic field is generated by three categories of sources: interior sources $\hat{f}^+$, monopole-type control inputs $\hat{g}$ along the perimeter $r = R$, or in other words, as we are discussing the Fourier representation, control function $\hat{g}(r) = A\delta(r - R)$ with the pointwise support at $r = R$, and exterior sources $\hat{f}^-$. The responses to these inputs will be $\hat{u}^+$, $\hat{v}$, and $\hat{u}^-$, respectively. The governing equation is (1.5). In Fourier representation, the total response is denoted by $\hat{u} = \hat{u}^+ + \hat{v} + \hat{u}^-$. Note that in the general discussion of section 2 we have also first obtained the annihilating function $v(x)$ as a response to only surface excitation (see formulae (2.10), (2.12), (2.13)) and then recovered the same function as an output of the volumetric control (2.20) constructed with the help of the auxiliary function $w(x)$. In the framework of the current example, we analyze volumetric controls in section 5.9.

The interior region can be completely shielded from the exterior noise by devising control inputs which exactly compensate for the exterior contribution to $\hat{u}$, such that (2.11) holds. This condition is equivalent to requiring that

$$\hat{u}^-(R) + \hat{v}(R) = 0 \tag{5.11}$$

and

$$\lim_{r\uparrow R} \left(\frac{d\hat{u}^-}{dr} + \frac{d\hat{v}}{dr} \right) = 0. \tag{5.12}$$

Both $\hat{v}$ and $\hat{u}^-$ are responses to exterior sources relative to the open domain Ω, so both $\hat{v}$ and $\hat{u}^-$ separately satisfy (5.9b). Since (5.9b) and (5.11) imply (5.12), the condition (5.12) is redundant. Therefore, (5.11) is a sufficient condition which we will use. Integral (5.6) along with the representation $\hat{g}(r) = A\delta(r - R)$ for the control, in which the constant A is yet to be determined, immediately yields that, for $r < R$, $\hat{v}(r) = A \cdot 2\pi R \cdot \hat{G}(r, R)$, and using the continuity of $\hat{v}(r)$ and equality (5.11), we conclude that $A = -\hat{u}^-(R)/(2\pi R \cdot \hat{G}(R, R))$. Consequently, the control input is given by the formula

$$\hat{g}(r) = -\frac{4\hat{u}^-(R)}{J_m(kR)(Y_m(kR) + iJ_m(kR))} \frac{\delta(r - R)}{2\pi R} , \tag{5.13a}$$

which can be evaluated using (5.10c) and simplified to read

$$\hat{g}(r) = \left\{ \frac{d\hat{u}}{dr} - \hat{u}\frac{d\log(Y_m(kr) + iJ_m(kr))}{dr} \right\}\bigg|_{r=R} \delta(r - R). \tag{5.13b}$$

In compliance with the general theory developed in sections 2 and 3, expression (5.13b) requires the knowledge of the total $\hat{u}$ and $d\hat{u}/dr$ along the *exterior* of the perimeter $r = R$ but does not require any information about the nature of exterior sound sources. Moreover, the control (5.13b) exactly shields the interior region $r < R$ from the exterior noise, while leaving the interior sound component completely unaffected. Realizable approximations of this exact solution may prove an effective method of reducing exterior noise.

5.7. Control of a single exterior source. As has been shown in sections 2 and 3, and also follows from the previous analysis in section 5.5, the control $\hat{g}$ given by (5.13b) is insensitive to the contribution of interior sources to $\hat{u}$ and $\frac{d\hat{u}}{dr}$. Let us consider a single exterior source located at $s > R$ whose forcing function $\hat{f}$ is given by (5.4). For this exterior source the response function is $\hat{u}^-(R) = \hat{G}(R, s)$ (see (5.6)), and substituting this expression into (5.13a), we obtain

$$\hat{g}(r) = -\frac{Y_m(ks) + iJ_m(ks)}{Y_m(kR) + iJ_m(kR)} \frac{\delta(r - R)}{2\pi R} . \tag{5.14}$$

Knowing the asymptotic behavior of Bessel functions as $m \longrightarrow \infty$, we can immediately see that $\hat{g} \equiv \hat{g}_m \sim \left(\frac{R}{s}\right)^m \frac{\delta(r-R)}{2\pi R}$, and therefore when $s > R$, the Fourier series with the coefficients $\hat{g}_m$ is convergent to an infinitely smooth function of the argument θ.
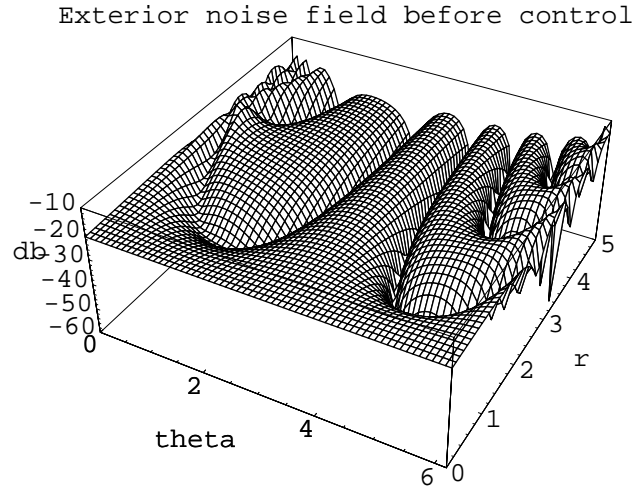
Exterior noise field before control

FIG. 5.1. *A particular sound field $u^-(r,\theta)$ generated by the exterior sources given by formula (5.15). The vertical scale is in decibels.*
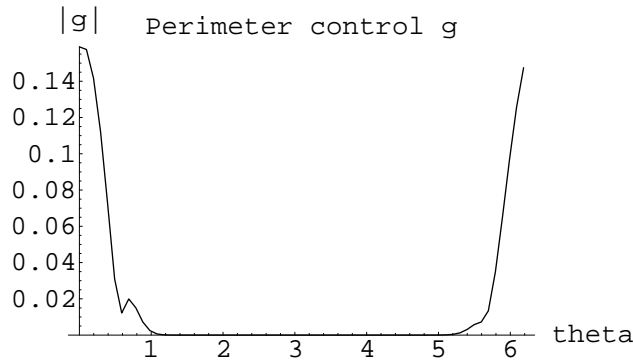

FIG. 5.2. *Control effort $|g|$ along the perimeter $r = 1$.*

5.8. A computational example. As a computational example, we have considered the problem of shielding a domain of radius $R = 1$ from the time-harmonic acoustic disturbances with the wavenumber $k = 5$ (see (1.1)). A particular exterior noise field u^- was generated by the forcing function

(5.15)
$$f^-(x,y) = \delta(x-3)\delta(y-1)$$
$$-\delta(x-5)\delta(y+1)$$

which produces the noise field u^- depicted in Figure 5.1.

Figure 5.2 shows the required control input along the perimeter, which can be computed using (5.13b).

This perimeter input produces a control acoustic field v depicted in Figure 5.3.

With control applied, the noise level within the protected region $r < R = 1$ is reduced to zero. This can be seen in Figure 5.4.
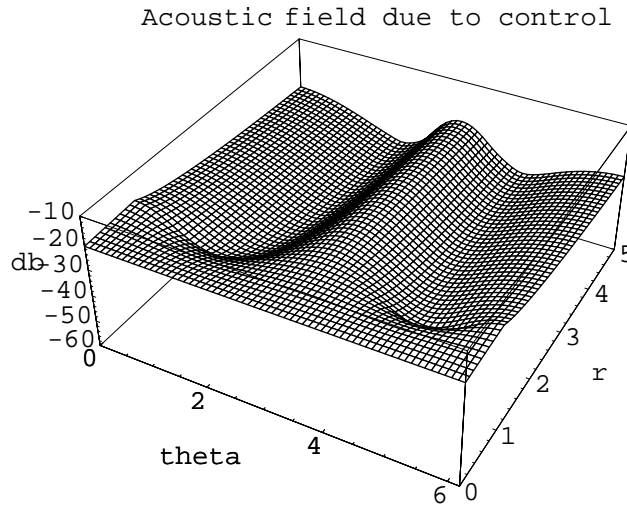
Acoustic field due to control

FIG. 5.3. *The sound field $v(r,\theta)$ generated by the control input along the perimeter. The vertical scale is in decibels.*
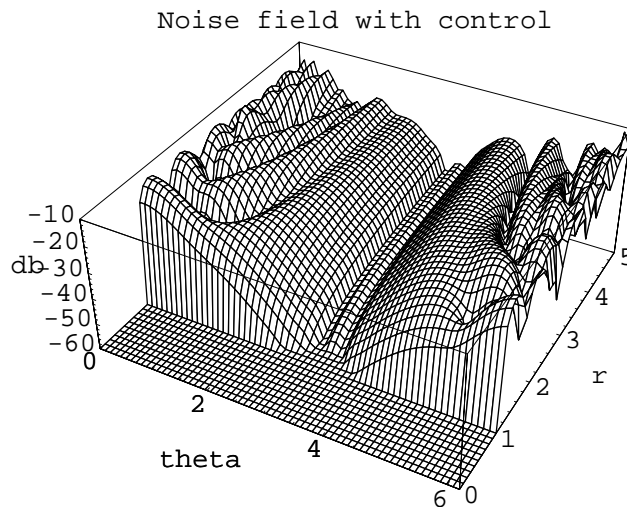
Noise field with control

FIG. 5.4. *The sound field $u^-(r,\theta)+v(r,\theta)$ resulting after the perimeter control has been applied. The vertical scale is in decibels.*

We note that in Figures 5.1, 5.3, and 5.4 the vertical scale is given in decibels; these units are defined as db $= 20\log_{10}|A|$, where A is the amplitude of a signal, or alternatively, db $= 10\log_{10}|A|^2$, where $|A|^2$ is proportional to power.

5.9. Optimal control with annular support. As opposed to section 5.6, in which we analyzed surface controls, here we consider volumetric control sources compactly supported on the annular region of thickness a. In other words, we now have interior sources in the region $r < R$, exterior sources in $r \geq R$, and control sources within $R \leq r \leq R + a$. Our goal will be to find a control which cancels out the unwanted exterior noise on Ω and is also optimal in the sense of a cost criterion

specified below (see (5.17)).

We note that according to Proposition 2.1 the entire variety of appropriate controls is described by formula (2.20), where the auxiliary function $w(x)$ satisfies the Sommerfeld condition (2.2a) and boundary conditions (2.14). To obtain compactly supported controls it is natural (as has been repeatedly pointed out in sections 2, 3, and 4) to consider compactly supported functions $w(x)$. However, this approach will not, generally speaking, yield all possible compactly supported controls because a function $w(x)$, which is not compactly supported near the boundary, may nonetheless generate a compactly supported $g(x)$ according to formula (2.20). Still, we expect that, on the one hand, for the problems that originate from applications, this approach will provide for the only tangible way to parameterize the compactly supported controls and, on the other hand, the set of controls obtained this way will be sufficiently wide for subsequent optimization.

For the purpose of the demonstration in this section it will be convenient to consider a different subclass of compactly supported controls, namely, square integrable controls. The simple formulation that we study here, i.e., the formulation that presumes the isotropy of the supporting medium everywhere and provides for the separation of variables along the boundary, will allow us to explicitly construct the optimal control in this class, where the optimality is treated as a minimum of the standard L_2 norm; see (5.17).

Using the same argument as before, we see that the interior will be exactly shielded from the exterior sources provided that the linear constraint (5.11) holds at the boundary $r = R$. This constraint written explicitly is

$$(5.16) \qquad \hat{u}^-(R) + \int_R^{R+a} \hat{G}(R,s)\hat{g}(s)\, 2\pi s \, ds = 0,$$

which is satisfied by many alternative distributions of control sources. Subject to this linear constraint, we shall seek to minimize the cost function defined by the L_2 norm of the control effort:

$$(5.17) \qquad \|\hat{g}\|^2 = \int_R^{R+a} |\hat{g}(s)|^2 \, 2\pi s \, ds \longrightarrow \min.$$

Let us consider the space of square integrable control inputs $\hat{g} : [R, R+a] \to \mathbb{C}$. At the optimal solution $\hat{g}$, which is a point in this space, the corresponding level set of the control effort norm must be tangent to the hyperplane of complex codimension 1 defined by the linear constraint (5.16). If we perturb the optimal $\hat{g}$ by a perturbation γ within the constraint set, the cost must never decrease:

$$\|\hat{g} + \gamma\|^2 = \int_R^{R+a} (|\hat{g}(s)|^2 + |\gamma(s)|^2 + \overline{\hat{g}(s)}\gamma(s) + \hat{g}(s)\overline{\gamma(s)})\, 2\pi s \, ds \geq \|\hat{g}\|^2.$$

This can happen only if *all* allowable perturbations γ satisfy the following (real) orthogonality condition:

$$(5.18) \qquad \langle \hat{g}, \gamma \rangle \overset{\text{def}}{=} \frac{1}{2} \int_R^{R+a} (\overline{\hat{g}(s)}\gamma(s) + \hat{g}(s)\overline{\gamma(s)})\, 2\pi s \, ds = 0.$$

Since the perturbation γ is limited only by the single complex constraint (obtained as a variation of (5.16)):

$$(5.19) \qquad \int_R^{R+a} \hat{G}(R,s)\gamma(s)\, 2\pi s \, ds = 0,$$
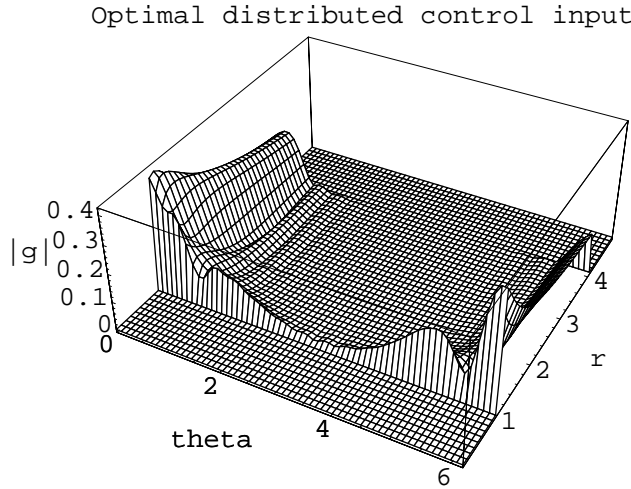
Optimal distributed control input

FIG. 5.5. *The magnitude of the optimal distributed control input $|g(r,\theta)|$ with support $1 \le r \le 4$ for the computational example of section 5.8 with noise sources given by formula (5.15).*

the orthogonality condition (5.18) imposes a severe limit on the form of $\hat{g}$. We note that (5.19) implies that all allowed perturbations $\gamma(s)$ are orthogonal to all functions of the type $p(s) = C \cdot \overline{\hat{G}(R,s)}$, where C is a complex constant, in the sense of (5.18):

$$(5.20) \quad \langle p, \gamma \rangle = \frac{1}{2} \int_R^{R+a} (\overline{C}\hat{G}(R,s)\gamma(s) + C\overline{\hat{G}(R,s)\gamma(s)}) \, 2\pi s \, ds = 0 \qquad \forall C \in \mathbb{C},$$

because the integral in (5.20) is the real part of the integral in (5.19) premultiplied by $\overline{C}$. Since C is arbitrary, the converse also holds, so that conditions (5.19) and (5.20) are equivalent. Therefore, the complex one-dimensional subspace of functions $\{p(r)\} = \{C \cdot \overline{\hat{G}(R,r)} \,|\, C \in \mathbb{C}\}$ spanned by the function $\overline{\hat{G}(R,r)}$ is orthogonal to all allowed perturbations. It follows from (5.18) that the optimal control $\hat{g}$ must lie within the subspace $\{p\}$. Therefore, within the annular support of the control input $R \le r \le R+a$, $\hat{g}(r) = C \cdot \overline{\hat{G}(R,r)}$ for some complex constant C (elsewhere, $\hat{g}(r) \equiv 0$). As the last step, the constant of proportionality C is obtained from (5.16). The unique solution for the control that we obtain by this method is given by

(5.21)

$$\hat{g}(r) = -\frac{\hat{u}^-(R)\,\overline{\hat{G}(R,r)}}{\displaystyle\int_R^{R+a} |\hat{G}(R,s)|^2\, 2\pi s \, ds}$$

$$= -\frac{4\hat{u}^-(R)\,(Y_m(kr) - iJ_m(kr))}{\pi J_m(kR)\Big\{s^2\big[J_m(ks)^2 - J_{m-1}(ks)J_{m+1}(ks) + Y_m(ks)^2 - Y_{m-1}(ks)Y_{m+1}(ks)\big]\Big\}\Big|_{s=R}^{s=R+a}}$$

for $R \le r \le R+a$, and $\hat{g}(r) = 0$ otherwise. The term $\hat{u}^-(R)$ in (5.21) must be evaluated via (5.10c) which requires the measurement of $\hat{u}$ and $d\hat{u}/dr$ along the perimeter $r = R$ *prior* to the application of the control. Alternatively, this process of measurement and control can be iterated in order to adapt to slow changes in the noise field. The minimum cost of this optimal control is $\|\hat{g}\|^2 = |\hat{u}^-(R)|^2$.

Returning to the computational example of section 5.8, the optimal distributed control input to exterior noise sources given by formula (5.15) was computed using (5.21). In this example we used $R = 1$ and $a = 3$. The amplitude $|g|$ of the resulting optimal control is depicted in Figure 5.5.

6. Discussion and conclusions. We have presented an accurate mathematical formulation of the problem of active shielding of a predetermined region of space from time-harmonic acoustic disturbances. We have constructed a general solution of this problem in the closed form and, using the apparatus of generalized Calderon's potentials and boundary projection operators, analyzed several consecutively more complex cases: the "pure" Helmholtz equation on the entire space with the Sommerfeld boundary conditions at infinity, other types of the homogeneous far-field boundary conditions that may be set either at infinity or at a finite external boundary, spatial anisotropies and discontinuities in the material properties, and certain kinds of nonlinearities.
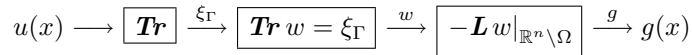
$$u(x) \longrightarrow \boxed{\boldsymbol{Tr}} \xrightarrow{\xi_\Gamma} \boxed{\boldsymbol{Tr}\, w = \xi_\Gamma} \xrightarrow{w} \boxed{-\boldsymbol{L}\, w|_{\mathbb{R}^n \setminus \Omega}} \xrightarrow{g} g(x)$$

FIG. 6.1. *Block diagram illustrating the information flow. The acoustic field $u(x)$ and its normal derivative are sensed along the perimeter Γ, providing the trace ξ_Γ. Each noise-canceling control input $g(x)$ corresponds to some function w, which is subject to only a few constraints. Since a large family of functions w will lead to exact volumetric noise cancellation, a particular w can be chosen by imposing additional requirements, e.g., by optimizing some criterion.*

Our approach to the problem of active noise control, which is schematically illustrated in Figure 6.1, possesses several key advantages. It does not require any knowledge of either structure or location or strength of the actual sources of noise that is about to be canceled, nor does it require knowledge of the properties of the medium across which the acoustic signals propagate, except, maybe, right next to the boundary of the domain to be shielded. It guarantees the exact volumetric cancellation of the unwanted noise throughout the domain of interest, while the input information sufficient for building the controls is given by a particular set of measurements performed only at the perimeter of this domain. Moreover, the control sources themselves also are concentrated only on or near the perimeter of the region to be shielded, thus rendering an effective surface control of the volumetric properties. Finally, the controls are constructed to eliminate only the component of the acoustic field generated by the exterior sources (i.e., located outside the domain of interest), while leaving the component that comes from the interior sources completely unaffected. In so doing, however, the measurements performed on the perimeter of the domain can refer to the total acoustic field rather than its unwanted exterior component only, and the methodology can automatically tell between the interior and exterior contributions to the overall field.

To demonstrate that the technique is appropriate, we have thoroughly analyzed a model two-dimensional example, and using the separation of variables in the Helmholtz equation written in polar coordinates and the apparatus of Bessel functions, constructed both purely surface and near-surface volumetric controls for shielding a cylindrical region from a given distribution of the noise sources.

In the future, the discrete framework for the active noise control is going to be used for analyzing complex configurations that originate from practical designs. This discrete framework has, in fact, already been developed to a large extent (see [19, 20, 21, 22]). It possesses the same set of attractive features as the foregoing

continuous formulation. The discrete formulation is based on the difference potentials method by Ryaben'kii [28, 29, 30] and uses finite-difference analogues of Calderon's potentials and boundary projection for constructing the controls and analyzing their properties. As opposed to the continuous model described in this paper, in the finite-difference framework both the measurements are performed and the control sources are located at the grid nodes, i.e., there are discrete sets of sensors and actuators for the acoustic field, which is obviously more feasible from the standpoints of physics and engineering applications.

There are most important issues yet to be addressed, in particular in the discrete framework, aimed at eventually creating practical designs. First, there are considerations of conditioning—specifically, how the measurement errors that are inevitable in any practically engineered system of sensors will propagate through the control system. Second, there are considerations of optimality. As we have seen, there is an entire family of control functions that can eliminate the unwanted noise on a given domain—this is, in fact, true for both continuous and discrete formulations. For a simple model example in the continuous formulation, we have shown in section 5.9 how to find a particular representative of this family, which is optimal in the sense of a certain criterion. Generally, optimizing the general solution for controls is a separate substantial task composed of a number of subproblems.

Namely, the criteria for optimization (i.e., objective functions) that would fit different practical requirements need to be clearly identified. These criteria will certainly be problem-dependent. For example, the designer of a noise suppression system in the passenger compartment of an aircraft should obviously try to minimize its weight and energy consumption, while for suppressing household appliances' noise the primary concern may be the cost of the active control system. After identifying the objective(s) for optimization, an appropriate optimization methodology has to be chosen; it may be either gradient-based or combinatorial (or a combination of the two) and may also include the initial effectiveness filters. Moreover, there may be different levels for optimization. In the beginning, we can search through the family of exact solutions to the noise control problem that is at our disposal (see sections 2, 3, and 4) and thus obtain the one that will be optimal in some sense. In so doing, we are still guaranteed that whatever optimum we find will still do the job, or in other words, exactly cancel out the unwanted noise on the domain of interest. If, however, the optimum we find this way is still unsatisfactory from the standpoint of the chosen criteria, we can go beyond the optimization of the exact solution only. In fact, we can introduce the tolerance, i.e., acceptable level of noise reduction as opposed to the exact cancellation, and using this greater flexibility try to better satisfy the optimization criteria. This is already an approximate optimization because it deviates from the exact solution for controls; as such, it will certainly require solving the finite-difference governing equation(s) repeatedly inside the optimization loop for the purpose of assessing the quality of the noise reduction. (Again, in the previous case the noise cancellation is exact and this is known ahead of time.) A prerequisite of this optimization, of course, is an efficient analysis code; most likely, fast parallel codes will be needed for solving real-life problems.

In the paper, we have addressed only the case of time-harmonic disturbances. In fact, the methodology will work for quasi-stationary acoustic fields as well, i.e., those that change slowly in time (slowly on the scale of the inverse frequency of time-harmonic oscillations). In the latter case, practical implementation of the noise suppression technique will require real-time measurements. If, however, we are dealing

with stationary interior and exterior fields, we can use the simplified version of the technique that permits precalculation of the dependence of the response on domain geometry and system properties. In other words, a microphone is needed in the final system only to sense phase and amplitude. Together with this, preprogrammed noise-canceling actuators are sufficient.

Finally, we should mention that time-harmonic (i.e., Helmholtz) problems of active shielding (as well as the aforementioned quasi-stationary problems) represent only a portion of the overall variety of formulations that the potential users of such methodologies would like to have explored. Even though these Helmholtz problems are of a substantial significance themselves from the standpoints of both mathematics and applications, it is clear that true time-dependent problems, i.e., those that accommodate broad-band acoustic fields, will ultimately be demanded by practitioners and therefore deserve theoretical and numerical study. The problem of active noise control in the formulation that involves broad-band spectra of frequencies was studied by Malyuzhinets [35] and Fedoryuk [36]. Some initial results on the time-dependent active shielding problem obtained in the framework of generalized Calderon's potentials can be found in work by Zinoviev and Ryaben'kii [37]. As has been mentioned in section 3.4, we expect that unsteady versions of Calderon's potentials and projections (see, e.g., [32, 33]) will help us approach the time-dependent noise control problems in much the same way as conventional Calderon's theory has been applied to time-harmonic noise control. Generally, the extension to broad-band spectra of disturbances is nontrivial in every respect and provides a novel challenge from the standpoints of both theoretical analysis and subsequent practical implementation.

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