MATHEMATICAL LIFE

Viktor Solomonovich Ryaben'kii and his school (on his 90th birthday)

Professor Viktor Solomonovich Ryaben'kii, doctor of the physical and mathematical sciences and a prominent expert in computational mathematics, observed his 90th birthday on 20 March 2013.

He was born in Moscow, in the family of state employees Solomon Abramovich Ryaben'kii and Berta Pavlovna Ryaben'kaya. In 1940 he enrolled in the Faculty of Mechanics and Mathematics of Moscow State University, but World War II interrupted his studies. After many narrow escapes from death, the mechanic and driver Ryaben'kii greeted Victory Day as a sergeant in the Guards, returned to his dear Faculty, graduated in 1949, and was admitted for graduate studies. The talents of the future leading figure of Russian computational mathematics were recognized and supported by Ivan Georgievich Petrovsky, the prominent mathematician and administrator of the sciences, who supervised Ryaben'kii's work first for his diploma thesis and then for his Ph.D. dissertation. This dissertation, under the long and — for that time — highly specialized title "Stability of finite-difference schemes and the application of the method of finite differences to solution of the Cauchy problem for systems of partial differential equations", defended at Moscow State University in 1952, revealed a new name to the computation community, and over time justified inscribing Ryaben'kii's name in the list of illustrious founders of the theory of finite-difference schemes.

It so happened that the young specialist Ryaben'kii began his career at about the same time as the (then secret) Department of Applied Mathematics was formed, initially as a division of the Mathematical Institute. Mstislav Vsevoldovich Keldysh, a brilliant mathematician, an expert in mechanics, and a statesman (who subsequently became President of the Academy of Sciences (1961–1975) at the age of 50) was its head. The Department of Applied Mathematics was organized to solve computational problems in nuclear and thermonuclear power generation,

outer space research, and other areas, and it was expanding by enlisting young talented researchers. Keldysh hired only people who could likely cope with problems that were absolutely new for physics and mathematics. Among those who attracted notice was Ryaben'kii, who then worked at the All-Union Distant Education Institute of Railway Engineering. In 1957, on the recommendation of Sergei Vsevoldovich Yablonskii, he was invited to the Computation Bureau of the Department. It was at the Keldysh Institute of Applied Mathematics (the current name of the Department of Applied Mathematics) that Ryaben'kii eventually developed into an internationally recognized researcher, who to this day continues his fruitful scientific investigations.

As a development of his Ph.D. thesis, Ryaben'kii wrote the book [1] in conjunction with Aleksei Fedorovich Filippov, his colleague of the same age, who subsequently became a well-known expert in differential equations. Acknowledged now as a classical treatise, it was in fact the first book on stability of finite-difference schemes in the world literature. Computational mathematics — a field as ancient as mathematics itself — had its second birth at that time. Large-scale calculations involving 'finite-difference schemes' for solving problems related to nuclear projects, the conquest of space, aircraft construction, and so on, and carried out with electromechanical desktop calculating machines or first-generation computers led to a revision and reappraisal of the then established ideas about methods for approximate calculations. Grids in a space could contain dozens or even hundreds of nodes (now even grids with 10^{10} points are not unusual), and appropriate theoretical bases for computational algorithms had to be developed. A real headache for people doing calculations was (and still is) the instability of numerical solutions which manifests itself as a rapid non-physical growth of the function being calculated at a point or simultaneously at a number of points, in dynamical problems as well as in steady-state problems being solved by relaxation methods. For this reason the methods and approaches proposed in [1] were most welcome to the experts in the field. It should be noted that at that time Ryaben'kii (and apparently also Filippov) was not at all occupied with calculations for practical purposes: as a 'pure' mathematician, he regarded finite-difference schemes as an abstract model and managed to determine and investigate all the main features of the behaviour of the model. This is a shining example of the deep innate mathematical intuition that pervades the whole of Ryaben'kii's research.

Of course, others had investigated difference equations well before [1]. For instance, the remarkable 1928 paper [2] by Courant, Friedrichs, and Lewy had been translated in *Uspekhi Matematicheskikh Nauk* before the war. But these were isolated, 'not fashionable' publications, partly because practical calculations and the use of the first computers were classified topics, to which only a very restricted circle of experts had access. Perhaps this is why the brilliant mathematician Peter Lax, who in the 1940s and 50s worked occasionally on the Manhattan Project at the Los Alamos National Laboratory and who also investigated questions on solution of boundary-value problems using finite-difference approximations, published his theory only in 1956 [3]. One of the main results in [1] and [3], which in the professional slang easily remembered by students, can be expressed by the phrase "approxima-

tion and stability yield convergence", now has several names: Ryaben'kii's theorem, the Ryaben'kii–Filippov theorem, Lax's theorem, and the Lax–Ryaben'kii theorem.

It should be noted that several teams of researchers were formed at that time in the Institute of Applied Mathematic to investigate a number of different 'classified' applied problems often requiring the solution of very similar theoretical questions subsequently forming the bases of many numerical methods. The communication of non-classified results and ardent discussions took place at research seminars and during dissertation defences. This undoubtedly contributed to an atmosphere of competition, enhanced by the fact that one of the departments of the institute, where stability of finite-difference schemes was also being investigated, had Aleksandr Andreevich Samarskii as its head, a brilliant researcher, a future member of the Russian Academy of Sciences, an organizer of science, and a prominent expert in computational mathematics, mathematical physics, and the theory of mathematical modelling. Ryaben'kii and Samarskii developed a long and trusting relationship of two experts who valued the opinions of each other, especially when it came to research results. Perhaps this had something to do with the fact that both had volunteered for the front as students and, after fate let them survive the war, had returned to their dear university. Some people recall Ryaben'kii talking about how, already as well-established academics, they would sometimes inquire, on meeting at the canteen or in the hallway of the institute, about the sign of the other's dp over dt derivative, where p denoted the size of the 'paunch', while t was, of course, the time.

During Ryaben'kii's first years at the Keldysh Institute he worked closely with his very talented colleague Sergei Konstantinovich Godunov, a future member of the Russian Academy of Sciences and a prominent mathematician and expert in mechanics. They wanted to produce a theoretical exposition and a compilation of the experience accumulated in several departments of the institute in the numerical solution of complicated problems in mathematical physics. Their collaboration resulted in the monograph [4], which also became widely known. The revised and augmented edition [5] remains even today one of the basic introductions to the subject. From the standpoint of science, their theory is mainly focused on the fundamental concept of stability of computational algorithms. Beginning with concise mathematical definitions of approximation and stability and explaining their meaning by examples, the authors developed a system of classification, selection, and methods for constructing finite-difference schemes in typical problems in mathematical physics. Furthermore, besides the already 'traditional' stability analysis in the spirit of the Courant–Friedrichs–Lewy test, they presented an analysis of the asymptotic accumulation of calculation errors in finite-difference evolutionary schemes, since even then the number of operations in practical calculations was already so large that these asymptotic properties had a crucial influence on the applicability of various algorithms. Another distinctive feature was an analysis of the influence of boundary conditions on the stability of finite-difference schemes. These difficult and subtle investigations were initially motivated by results of the remarkable mathematicians Israel Moiseevich Gelfand and Konstantin Ivanovich Babenko that were presented at a conference in 1956 and that foreshadowed the future theory. However, a consistent implementation of their ideas turned out to be

a difficult task. Only after the introduction of some new and unusual notions such as the 'spectrum of a family of operators' (which emerged in discussions with Gelfand, Babenko, and the outstanding mathematician Èmmanuil Èl'evich Shnol' [6]) and then also the 'kernel of the spectrum of a family of operators' [5], could Godunov and Ryaben'kii make their theory harmonious and constructive and obtain results in closed form.

It should be pointed out that before [5] was written, it took both authors quite a long period of time (about 7 years) to 'harmonize' their points of view. This was a time of intensive joint work, when they overcame many differences in their understanding of various experimental tricks that had been used already for several years in numerous calculations of very diverse physical phenomena. The point was that models of these phenomena could not always be put in the form of well-posed mathematical problems. In such cases they had to look for a compromise by simplifying slightly the problems to be solved or by setting aside problems for which there were not yet concise mathematical formulations. The future development of concepts sketched at that time could be entirely unexpected. For example, the concept of the spectrum of a family of difference operators subsequently prompted the introduction of the concept of the spectral portrait of a matrix [7].[1]

The question of the accuracy of difference approximations of generalized discontinuous solutions of non-linear hyperbolic equations (for instance, in the calculation of shock waves in gas dynamics) has always been and still is crucial for the theory of finite-difference schemes. In 1958 Godunov and Ryaben'kii (with the help of the calculator Natal'ya Mikhailovna Goman'kova) performed a series of numerical experiments to finally resolve this question. Godunov recalls:

"These experiments produced results which baffled us. My difference scheme with decay of discontinuities,[2] which has first-order accuracy $O(h)$ on smooth solutions, could only ensure an error of order $O(\sqrt{h}\,)$ in computations of 'generalized' solutions with shock waves. We announced this at a scientific conference at Moscow State University, but at that time we had not had the opportunity to present our experiments as an article. Nevertheless, the interest in accuracy analysis in this kind of calculation was increasing anew and intensified in the 1970s, when many different numerical methods with enhanced accuracy appeared. I knew many of the creators of these methods, but in my discussions with them I could not find out just what they meant by 'order of accuracy', whereas in our investigations we followed the example of Sergei L'vovich Sobolev and used the concept of weak convergence. All these investigations of high-resolution difference schemes and my meetings with their authors were going at a time when I was occupied with quite different questions, so I did not delve into the details of their methods. In 1997 I was invited to the University of Michigan, where I gave the lecture 'Reminiscences about difference schemes'. When I was preparing it, the Novosibirsk mathematician Vladimir Viktorovich Ostapenko reproduced, at my request, my joint investigation with Ryaben'kii applied to the Harten–Lax high-resolution scheme, and thereby demonstrated that from our point of view it does not deliver higher accuracy. I presented the text of my lecture as a preprint in Russian and was assured that it would

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[1]See also [8], where similar concepts developed independently are described.

[2]*Russian editor's note*: The classical 'Godunov scheme'.

be translated and published in the *Journal of Computational Physics*. In fact, it was published two years later, but the text was abridged and the counterexamples constructed by Ostapenko were omitted.[3] In 2011, together with M. Nazar'eva and Yu. Manuzina, who were preparing their master's theses under my supervision, I carried out an analysis of weak convergence in numerical experiments based on my classical scheme, that is, I reproduced thoroughly my joint investigations with Ryaben'kii from 1958, taking a significantly larger set of examples. Again we obtained lower rates of weak convergence. Our results [9] were met without criticism, although we were prepared for stormy discussions. Recently I have found some colleagues in Novosibirsk, some of whom are performing numerous very delicate calculations using contemporary high-resolution schemes, while others have been modelling elastoplastic processes together with me. We have been able to organize a broad discussion of questions relating to accuracy and the organization of corresponding numerical experiments. It seems that we are now feeling our way forward through all the complexities in our understanding of the problem, and we believe we will be able to work out our points of view to the level of a publication. Hopefully, my remaining life time will be enough for me to be among the authors of this publication, thus completing the analysis of the problems which Ryaben'kii and I came across in the late 1950s."

Ryaben'kii made an interesting contribution to interpolation problems. To investigate the stability of systems of difference equations with respect to the initial data, he constructed an algorithm for local polynomial interpolation of a grid function with prescribed smoothness. Next, on the basis of this algorithm he developed a method for smooth local interpolation on non-uniform rectangular grids, a method now known as 'Ryaben'kii local splines' and used in quite a few computational algorithms and by theorists.

In completing the description of Ryaben'kii's research of the late 1950s to early 1960s, we can say that in essence the monographs [1], [4], and [5] have determined the main directions in the theoretical analysis of finite-difference schemes, a central object of computational mathematics and an important tool of contemporary mathematical modelling. Another interesting feature of the textbook [5] is its bibliographical commentary, which explains the origins of many lines of research that have developed into classical areas of computational mathematics.

The next stage of Ryaben'kii's research, continuing at present, gave computational mathematics the new concept of a difference potential together with a whole gamut of applications of these potentials, from numerical methods for solving boundary-value problems in mathematical physics to algorithms for active noise shielding. In what follows, in presenting the main landmarks of the development of the theory of difference potentials we will, of course, impose reasonable limits on the formal rigour of our presentation. The interested reader can find all the details in the papers and monographs cited below.

The story began with Ryaben'kii's doctoral dissertation "Some questions in the theory of difference boundary-value problems" [10], which he defended in 1969. Taking an arbitrary difference operator A with constant coefficients (for example,

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[3]The full English translation of the lecture was published in October 2008 in France, as INRIA preprint no. 6666 "Reminiscences about numerical schemes".

from an approximation of an elliptic differential operator) as a point of departure, Ryaben'kii introduced the notion of a (multilayer) *grid boundary* γ of a grid domain M and constructed certain objects Pu_γ which were entirely new for difference problems, namely, convolution sums of the finite-difference fundamental solution for the operator A with grid functions u_γ considered on γ (here and below we use a subscript to denote the trace of a function on a set S, that is, $u_S \equiv u\big|_S$). The formulae which he introduced turned out to be finite-difference analogues of Cauchy and Cauchy-type integrals, because they *a*) produce all the solutions of the homogeneous equation $Au_{\overline{M}} = 0_M$, where $\overline{M} = M \cup \gamma$, and *b*) define the *difference boundary projections* P_γ obtained by restricting P to the subspace of functions with support on γ: $P_\gamma u_\gamma \equiv (Pu_\gamma)_\gamma$. Thus, the shorthand expression $P_\gamma u_\gamma - u_\gamma = 0$, which he called the *inner boundary conditions*, is a complete difference analogue of the Sokhotskii–Plemelj integral relation (recall that the latter distinguishes the class of boundary functions that can be extended to the whole domain as analytic functions). Thanks to this relation he could in an equivalent way reduce a finite-difference boundary-value problem for A formulated in the domain $\overline{M}$ to grid equations on the boundary γ:

$$\begin{cases} P_\gamma u_\gamma - u_\gamma = \psi, \\ Bu_\gamma = \varphi, \end{cases} \tag{1}$$

where the second equation corresponds to the boundary conditions in the original difference boundary-value problem (for instance, to the Dirichlet conditions).

It will be no exaggeration to say that the operator P, which was subsequently called the *difference potential* operator and was generalized to the case of variable coefficients and evolution problems, became the central object of theoretical and applied investigations for Ryaben'kii and his school. These investigations involved questions of the algebraic formalism of the theory of difference potentials and classical potentials, the analysis of approximative properties of P, approaches to the economical computation of Pu_γ, methods for the efficient solution of various boundary-value problems in mathematical physics using a reduction of the form (1), and the use of ideas related to difference potentials in problems of constructing non-reflecting boundary conditions, active noise reduction, and many others. It should be noted that Ryaben'kii usually begins his lectures on the method of difference potentials as follows:

"The difference potential has its prototype in a Cauchy-type integral

$$f(z) = \frac{1}{2\pi i} \oint_\Gamma \frac{\rho_\Gamma(\zeta)}{\zeta - z}\, d\zeta, \qquad z \notin \Gamma,$$

defined on the space of piecewise smooth complex-valued functions with a jump on a closed contour Γ, which partitions the complex plane into a bounded domain Ω^+ and its complement. This integral can be treated as the potential of the Cauchy–Riemann differential operator $\partial/\partial\overline{z}$, where the function $\rho_\Gamma(\zeta)$, $\zeta \in \Gamma$, plays the part of the density of the potential. This potential (in contrast to single- and double-layer potentials for the Laplace, Helmholtz, Lamé, Stokes, Maxwell, and other operators) has a unique property: its construction involves the boundary

projections P_Γ^+ and P_Γ^- defined by

$$P_\Gamma^+ \rho_\Gamma(\zeta) := \frac{1}{2}\rho_\Gamma(\zeta) + \frac{1}{2\pi i} \oint_\Gamma \frac{\rho_\Gamma(\eta)}{\zeta - \eta}\, d\eta,$$

$$P_\Gamma^- \rho_\Gamma(\zeta) := \frac{1}{2}\rho_\Gamma(\zeta) - \frac{1}{2\pi i} \oint_\Gamma \frac{\rho_\Gamma(\eta)}{\zeta - \eta}\, d\eta.$$

The singular integrals are understood in the principal value sense.

Let $f(z)$ be a piecewise continuous analytic function tending to zero as $z \to \infty$ and having a jump on Γ. Also, let $f_\Gamma^+(\zeta)$ and $f_\Gamma^-(\zeta)$ be the limits (traces) of $f(z)$ as z approaches a point ζ on Γ from inside Ω^+ and from outside Ω^+, respectively. Then it is known that given the sum $f_\Gamma = f_\Gamma^+ + f_\Gamma^-$, the projections P_Γ^+ and P_Γ^- enable us to find each term from the formulae $f_\Gamma^+ = P_\Gamma^+ f_\Gamma$ and $f_\Gamma^- = P_\Gamma^- f_\Gamma$.

The potentials of general linear difference operators,[4] which are projections in the space of densities on the grid boundary, combine the unique properties of the Cauchy-type integral mentioned above with the universal applicability and algorithmicity of finite-difference schemes.

This is the cornerstone of most of the new opportunities opened by the *method of difference potentials*."

We return to the time when difference potentials were introduced. Ryaben'kii clearly realized what a conceptually rich and interesting new direction in computational mathematics had been opened for the investigation of the new objects, and he began attracting students of the Moscow Institute for Physics and Technology to his research. The first of them was A. Ya. Belyankov (diploma thesis of 1970 at the institute), who treated algebraic aspects of the construction of the operator P in his diploma thesis and subsequent papers, including his Ph.D. dissertation [11]. As a result, in conjunction with Ryaben'kii he constructed a difference analogue of the machinery of singular integral equations in the theory of finite-difference boundary-value problems. In particular, in [11] he gave an expression for difference potentials in the form

$$P u_{\overline{M}} := u_{\overline{M}} - \left(F(A u_{\overline{M}})_M\right)_{\overline{M}}, \tag{2}$$

where F is the difference Green operator for A in some larger domain $M_0 \supset \overline{M}$, that is, $F A u_{M_0} \equiv u_{M_0}$. This form has proved to be very convenient for the further development of the theory of difference potentials. One of its main properties is the fact that the result depends only on the values of the function $u_{\overline{M}}$ on the set $\gamma \subset \overline{M}$.

In Fig. 1 is an example of M, γ, and M_0 for a difference two-dimensional second-order elliptic operator A with the usual cross-shaped five-point stencil.

The theory of approximative properties of difference potentials under refinement of the mesh, sketched already in [10], was significantly developed by Ryaben'kii's next student, Aleksandr Anatol'evich Reznik (diploma thesis of 1979 at the institute).[5] Using the apparatus of generalized functions, he obtained the first results

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[4] *Russian editor's note*: Ryaben'kii potentials.

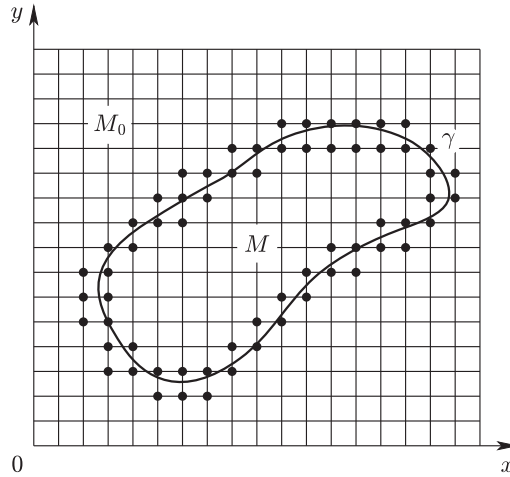[5] One of Ryaben'kii's most talented students, he sadly passed away while still young in 1995.

Figure 1.   The domains M (grid points inside the contour) and M_0 (grid square) and the boundary γ (bullets) for an operator A with a five-point stencil; $\overline{M} = M \cup \gamma$.

on approximation of the potentials of general elliptic operators in the norms of Hölder and Sobolev spaces [12], [13]. His approach to the approximation of elliptic surface potentials was based on several ideas. The first idea was Ryaben'kii's 'supplementary idea', announced in 1976 at a conference dedicated to the 75th birthday of Academician Petrovsky (see [5], Supplement, § 10), or more precisely, the part of that idea relating to the extension of the Cauchy data to the grid boundary. For definiteness let us consider a second-order elliptic operator L in a simply connected closed domain Ω with sufficiently smooth boundary Γ (for example, Γ can be the contour in Fig. 1). The extension algorithm works as follows: for a pair of functions $\{\nu, \partial\nu/\partial n\}$ that is the Cauchy data on Γ for some function ν, Taylor's formula is used to construct a function u_γ at points in the grid boundary γ corresponding to Γ:

$$u_a := \nu(b) + r\frac{\partial\nu}{\partial n}(b), \tag{3}$$

where $a \in \gamma$ is the current point in γ, $b \in \Gamma$ is the base of the normal dropped from a to Γ, and r is the length of the normal with the appropriate sign (depending on whether a is an exterior or interior point of Ω). It should be noted that with this definition of u_γ the meaning of multilayeredness of the boundary γ becomes intuitively clear: using functions on γ we can keep information about the entire vector of Cauchy data on Γ. Next, he used the idea of a unified representation for continuous and difference potentials in the form (2). For example, for the simple- and double-layer potentials

$$w_{\overline{\Omega}} = 0.5\rho_0 - \int_\Gamma \left(G\rho_1 - \frac{\partial G}{\partial n}\rho_0 \right) d\Gamma, \tag{4}$$

with densities ρ_1 and ρ_0 and the Green's function G, the Green's identity gives the representation

$$w_{\overline{\Omega}} = u_{\overline{\Omega}} - \big(G * (Lu_{\overline{\Omega}})_\Omega\big)_{\overline{\Omega}}, \qquad (5)$$

where u is an arbitrary function with the Cauchy data $\{\rho_0, \rho_1\}$ and $\overline{\Omega} = \Omega \cup \Gamma$. Finally, the third idea was that the operation of convolution $G * f$ can be approximated by solving the auxiliary difference problem

$$\begin{aligned} Au &= f_{M_0}, \\ u &\in U_0 \end{aligned} \qquad (6)$$

in the larger domain M_0, where U_0 is the subspace of functions on M_0 that satisfy a certain homogeneous boundary condition.

Remark 1. The subspace U_0 determines a particular Green's function in the set of all possible options: for instance, the homogeneous Dirichlet boundary condition corresponds to the Green's function of the Dirichlet problem in M_0, while we shall see below that more subtle conditions can correspond to the Green's function of the free space, the fundamental solution.

In fact, now we see how we can construct an algorithm approximating the potentials (4) at points in $\overline{M}$: first we use the extension formula (3) to construct the function u_γ on γ for the Cauchy data $\{\rho_0, \rho_1\}$, and then we use (2), shifting the main difficulties to the process for solving the problem (6); the function $u_{\overline{M}}$ is extended by zero outside γ. Thus, we miraculously avoid the need to know the explicit form of the Green's function G and to calculate the convolution with singular kernels in (4). In this way, by solving (6) using, say, the multigrid method due to Radii Petrovich Fedorenko [14] (who was a great friend of Ryaben'kii and his neighbour both in their apartment house and at work: their desks were in the same office) or another 'fast' numerical method, we can approximate the potentials of arbitrary elliptic operators with variable coefficients with the same degree of numerical efficiency. If in the above algorithm we confine ourselves to the second part, that is, to the formula (2), then we obtain just an algorithm for calculating difference potentials for an arbitrary grid function γ without knowing F.[6]

Remark 2. The approximative properties of difference potentials agree with the order of accuracy of the extension of the Cauchy data to γ. Clearly, the extension (3) is the simplest (zero-order) approximation of the solution of the Cauchy problem in a small neighbourhood of Γ for the equation $L\nu = 0$. The approximations next in order are obtained by adding new terms of the Taylor series, which are recursively calculated from the tangential derivatives of the Cauchy data using the conditions $\dfrac{\partial^k}{\partial n^k}(L\nu) = 0, \ k = 0, 1, 2, \ldots$. The parameter k determines the indices of the Hölder and Sobolev spaces in which the estimates of approximations are computed.

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[6]This construction has become a classical component of the theory of difference potentials. However, for instance, I. L. Sofronov recalls how in his student years, after managing to immerse himself deeply in the theory of inner boundary conditions and understand all the intricacies of the problem of finding difference fundamental solutions, he heard from the 'chief' that difference potentials could now be calculated without producing these solutions. This news made a very strong impression on him, and he almost had doubts about whether it could be true.

The importance of the above scheme for computing difference potentials and the importance of the method for approximating classical potentials can undoubtedly be compared with the importance of the very idea of difference potentials. It was due to the joint efforts of Ryaben'kii himself and his first students Belyankov and Reznik to solve these problems that difference potentials found their way into common practice in computations. It is a significant feature of these constructions that continuous potentials can be approximated algorithmically in a rather natural and routine way, and without having to use Green's functions or to calculate singular surface integrals.

The algebraic aspect of the construction of difference potentials is beautiful and very attractive. For more than 40 years since he thought of the operator P, Pyaben'kii has been repeatedly returning to this subject. He introduced the concept of the *clear trace* of a function u_γ. The clear trace allows one to control the parametrization of the set of difference potentials Pu_γ by means of the kernel of P (recall that P is a projection, whose kernel has dimension equal approximately to half the number of points in γ). His investigations in the early 1980s resulted in the construction of and the theory of difference potentials for general linear systems of difference equations on abstract grids. On the other hand, it turned out that this algebraic formalism could be carried over also to linear differential operators and boundary-value problems. Thus, the 'finite-difference' theory prompted the development of a similar 'continuous' theory (although Ryaben'kii was of course thinking about the two formalisms at the same time: the first publications on both subjects, [15] and [16], appeared in the same year 1983). This led to the creation of a general construction of surface potentials and projections for differential operators on the basis of the definition (5). It was presented in developed form in [17] (see also [18]) and covered the Plemelj–Sokhotskii equations, Calderón–Seeley boundary potentials (projections), classical potentials, and Green's formulae as special cases. Some time later M. I. Lazarev, Ryaben'kii's colleague and close friend, focused on the algebraic aspects of the notion of a clear trace, the notion of a potential with density in the space of clear traces, and the notion of boundary equations with projections, as they were considered in [17]. He defined a partial ordering in the set of all possible constructions of clear traces and defined the minimal clear trace [19]. Recently, Ryaben'kii returned to the algebra of difference potentials and proposed constructions enabling a parametrization of the whole family of difference potentials. These results are a further step along the road to formalizing the selection of a difference potential which is adequate to the aims of its use.

Now we look at some areas where the theory of difference potentials is applied.

1) Since they were introduced, the use of boundary potentials has been focused on boundary-value problems of mathematical physics. For many years Ryaben'kii worked together with his students and colleagues on the development of sufficiently 'universal' techniques which would enable them to get fast difference algorithms on rectangular grids for curvilinear domains, based on the reduction (1). In the early 1980s they finally developed such a scheme, tested it in several problems [13], [20], and published it in [21]. We will describe the main components and features of this

scheme using the example of the boundary-value problem

$$\begin{cases} Lu = 0 & \text{in } \Omega, \\ l\left(u, \dfrac{\partial u}{\partial n}\right) = f & \text{on } \Gamma, \end{cases} \tag{7}$$

where the operator L and the domain Ω with curvilinear boundary Γ are as introduced above, and l can be any boundary-value operator, provided that the problem (7) is well posed. We start by constructing the difference potential. To do this we put Ω in a parallelepiped Ω_0 and introduce a uniform rectangular grid in Ω_0. This is the domain M_0 on which we approximate L by an operator A. Using the formalism of difference potentials, we define the grid boundary γ and then the operator P by the formula (2). Further, we represent the unknown functions u and $\partial u/\partial n$ by their values at the points of some set (grid) S on Γ; let $\bar{d}$ be the vector of these values. Interpolating $\bar{d}$ to u and $\partial u/\partial n$ on the whole of Γ by splines, we use (3) to construct an operator Π extending the Cauchy data from Γ to γ: $u_\gamma = \Pi\bar{d}$ (see Remark 2 concerning the accuracy). Finally, introducing Euclidean norms U_γ and U_S on γ and S in terms of the sums of squares of the values of the functions and, if necessary, the sums of the squares of their first difference quotients, we state the discrete problem of minimizing the sum of two discrepancies by choosing an appropriate $\bar{d}$:

$$\left\|\Pi\bar{d} - P\Pi\bar{d}\right\|_{U_\gamma}^2 + \left\|l\big|_S \bar{d} - f_S\right\|_{U_S}^2 \to \min_{\bar{d}}. \tag{8}$$

The first term corresponds to the inner boundary conditions on γ (see (1)), and the second approximates the boundary conditions on Γ. Clearly, the problem (8) corresponds to an Euler–Lagrange equation, a system of linear equations with a self-adjoint matrix. Solving this system using the conjugate gradient method, we find the required values of u and $\partial u/\partial n$ on S.

Let us list the main properties of the above algorithm.

(i) It is indeed universal with respect to the boundary-value operator l: we need not invent separate methods each time for approximation on a rectangular grid for different types of l, separate methods to include these operators in the difference scheme in the interior of the domain or for an efficient solution of the resulting system of equations. For (8) the operator l is approximated at points in the original curvilinear boundary.

(ii) There is no need to adapt the rectangular grid to the curvilinear boundary, because the relation between functions on γ and S is realized in a unified way, by the constructive algorithm for extending the Cauchy data that was implemented in the construction of the operator Π.

(iii) The limiting problem (8) obtained by letting the step size of the grids in M_0 and S tend to zero is solvable if and only if the original problem (7) is. This is important, for instance, in the case of the Helmholtz equation. It is known that using the method of boundary integral equations to reduce boundary-value problems to equations on the boundary for the Helmholtz equation can lead to the situation when the resulting operator of the boundary integral equations has no inverse for some frequencies (so-called inner resonances), although the original boundary-value

problem is well solvable. In the case of equation (8), which, like boundary integral equations, reduces the original problem to equations on the boundary, there is no such problem: as already noted, the difference potential (2) is an approximation of the Calderón–Seeley potential, so (8) can keep the solvability properties of the original problem.

(iv) The behaviour of the condition number of the Euler–Lagrange equation for (8) when the step size of the grid decreases depends essentially on the norms U_γ and U_S. The theory of elliptic boundary-value problems suggests that we must take the derivatives into account in order to have 'correct' norms. For such a choice of norms we avoid the situation when the condition number, and therefore also the rate of convergence of the iterative method, depends on the step size.

(v) In the main, the time consumption and memory usage at each iteration are determined by the resources used for solving (6) in M_0, that is, they compare, for instance, to those for the multigrid method (if we do not use some even more efficient method).

Remark 3. Property (i) is directly and fairly deeply connected with the fact that the boundary γ of the grid domain is multilayered. In analyzing the strong and weak aspects of the above algorithm, Sofronov proposed another approach [22] to the solution of regular elliptic problems on rectangular grids in curvilinear domains using difference potentials. In it, the original elliptic problem is first transformed into an equation with a zero-order self-adjoint pseudodifferential operator acting in a Hilbert space which is a product of weighted Sobolev spaces. Then this operator is approximated on the rectangular grid, where the unknown functions are taken at points in $M \cup \gamma$; here difference potentials are used as preconditioners, which ensures the zero order of the corresponding self-adjoint discrete operator in a finite-dimensional Hilbert space.

2) The choice of the subspace U_0 on M_0 for the calculation of the difference potential (2) is a significant degree of freedom, which must be used wisely. When we discuss inner problems (Ω is bounded), the main criterion is a fast and reliable solution of (6); so, as we have already indicated, U_0 can, for instance, correspond to homogeneous Dirichlet conditions. However, for solving *outer problems* we need to construct U_0 so as to have the correct asymptotic behaviour of the difference potential at infinity. In 1982 Ryaben'kii interested his next student Sofronov (diploma thesis of 1981 at the Moscow Institute for Physics and Technology) in these problems. As a result, they found an appropriate difference potential and constructed an algorithm for solving outer problems for the Helmholtz equation [20], [23], based on Ryaben'kii's idea of using for the definition of U_0 the so-called *partial conditions* which are obtained via Hankel functions from the well-known representations of the general solution of the Helmholtz equation in a far field as Fourier series in spherical harmonics. In these constructions the domain Ω_0 is a sphere or a spherical shell, and the problem is stated in spherical coordinates.[7] The resulting difference potential approximates the solutions of the Helmholtz equation with Sommerfeld conditions

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[7]We remark that conditions of this type were later significantly developed by many authors and came to be called DtN (Dirichlet-to-Neumann) maps in the theory of non-reflecting artificial boundary conditions.

at infinity with the required accuracy, and its calculation is algorithmically just as simple as in the case of U_0 determined by the Dirichlet condition.

With the goal of extending these preliminary results on the construction of difference potentials for outer problems to the broadest possible classes of problems, Ryaben'kii subsequently developed the entirely original idea of constructing *difference operators for non-reflecting artificial boundary conditions* for steady-state problems on the basis of difference potentials. It is actually based on a well-known fact in the theory of difference potentials: if γ is the (open) outer boundary of the grid computational domain, then the equation $(Pu_\gamma)_\gamma - u_\gamma = 0$ with the operator P constructed from the difference fundamental solution will provide one of the possible forms for expressing the required non-reflecting artificial boundary conditions. These non-local relations for u_γ replace in an *equivalent way* the homogeneous equation for the original difference operator on the discarded infinite grid domain (that is, such non-reflecting artificial boundary conditions are *exact* in the finite-difference sense). Among important components of this circle of ideas we can name first of all an approach to an economical calculation of the vectors $(Pu_\gamma)_\gamma$ and, second, expressions for the resulting non-reflecting artificial boundary conditions which are more convenient for calculations. Ryaben'kii approached the realization of these ideas at the end of the 1980s, in joint papers with his new generation of students, S. V. Tsynkov (diploma thesis of 1989 at the Moscow Institute for Physics and Technology), M. N. Mishkov (diploma thesis of 1991 at the same institute), and V. A. Torgashev (diploma thesis of 1993 at the same institute). In [24]–[30] they developed techniques for the *approximate* calculation of the vectors Pu_γ with prescribed behaviour at infinity for various equations of mathematical physics. These techniques use the principle of *limiting absorption* by adding terms containing a small parameter to the original equations in order to single out the required asymptotic terms and reduce the size of M_0. As regards the expressions for non-reflecting artificial boundary conditions which can conveniently be used in algorithms solving the equations inside the computational domain, they can be obtained from the condition $(Pu_\gamma)_\gamma - u_\gamma = 0$, by partitioning the multilayer boundary γ into an outer layer and an inner layer and by constructing an operator interpolating the values of the solution on the outer layer from its values on the inner layer.

It should be noted that we need significant resources to construct the approximations required in the equation $(Pu_\gamma)_\gamma - u_\gamma = 0$. We can spare the resources needed for the calculation and use of the difference potentials if we find a convenient parametrization of the space of densities which will enable us to eliminate the kernel. D. S. Kamenetskii (diploma thesis of 1989 at the Moscow Institute for Physics and Technology), another student of Ryaben'kii, considered questions of a single-valued parametrization of the solution set of the general homogeneous difference equation by means of difference potentials with densities of various forms, and he investigated methods for constructing so-called independent inner boundary conditions and generalized Poincaré–Steklov difference operators [31], [32]. Also, Tsynkov's results in [33] are relevant here: there he used an example of a two-layer boundary γ (see Fig. 1) to express the densities of difference potentials which are

analogous to the densities of the single- and double-layer potentials for second-order differential operators.

It is an important argument for the use of operators of non-reflecting artificial boundary conditions constructed on the basis of difference potentials that they are usually meant for repeated computations in problems of the same type. These computations must reflect variations of the parameters (for example, the variation in the geometry of a body in diffraction or flow problems, the set of different right-hand sides, and so on) which affect the equations in the interior of the computational domain, but do not require new computations of the operators of non-reflecting artificial boundary conditions. Therefore, only the computational resources necessary for the use of the operators already constructed are important, and these are 'compensated with surplus' by the high accuracy of the non-reflecting artificial boundary conditions and the sharp reduction in the size of the computational domain (in comparison to the usually asymptotic boundary conditions at infinity).

An instructive example is given by Tsynkov's results in [34]–[39], where in problems of steady flows he used the approach proposed by himself and Ryaben'kii for constructing non-reflecting artificial boundary conditions on the basis of difference potentials. In designing the required non-reflecting artificial boundary conditions and realizing them using the codes FLOMG and TLNS3D developed by NASA for scientific and industrial calculations, he verified their efficiency in many test calculations for variations of the geometry of the objects placed in the flow (profiles, wings, and oblong bodies with jet propulsion), the Mach numbers of the incident flow, the size of the computational domain, the mesh of the grid, and so on. The two-dimensional code FLOMG is used for the integration of both the complete Navier–Stokes system and the system of equations in the thin layer approximation. The code TLNS3D was designed specifically for the thin layer equations. Both are based on finite-difference schemes with symmetric differences in the space variables and with artificial first- and third-order dissipation. The standard method for stating the far field conditions for both codes is to use certain local conditions on the outer boundary. This method is based on the assumption that the flow is 'almost one-dimensional' far away from the body, and also on a corresponding analysis of the input and output characteristics arising when time is introduced (a stationary solution is interpreted as a result of establishment). Tsynkov's experiments showed that the new non-reflecting artificial boundary conditions on the basis of difference potentials not only always reduce (up to a factor of three) the time required for calculations of the same accuracy, but also improve the stability of the calculations and produce stationary solutions in some cases when the calculations using the original codes have produced non-physical oscillations.

3) So far we have focused on elliptic equations of mathematical physics, although the theory of difference potentials also encompasses non-stationary problems. In the very beginning of the 1990s Ryaben'kii formulated the main constructions of non-reflecting artificial boundary conditions on the basis of difference potentials for explicit difference schemes [40] and suggested that Sofronov look at the case of a wave equation with constant coefficients in order to concretize the corresponding constructions. The first results of calculations of a difference fundamental solution using a standard second-order central-difference scheme were baffling: strong

oscillations, the absence of grid convergence in a neighbourhood of the front, and so on. To sort all this out, it was necessary to turn to analytic methods, which quite unexpectedly resulted in very different solutions of the problem, namely, to the development of analytic and quasi-analytic *transparent boundary conditions*.

The transparent boundary conditions proposed by Sofronov [41], [42] for the wave equation have the following form on the sphere and the circle ($d = 3, 2$):

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial r} + 0.5(d-1)\frac{u}{r} - Q^{-1}\{B_k^*\}Qu = 0,$$

where Q and Q^{-1} are the operators of the direct and inverse Fourier transformation in the spherical or trigonometric basis, and $\{B_k^*\}$ denotes a diagonal matrix of time-convolution operators that acts on the vector of Fourier coefficients (k is the index of a component). The convolution kernels $\mathrm{B}_k(t)$ are known and in practical applications are approximated by sums of exponentials:

$$\mathrm{B}_k(t) \approx \sum_{l=1}^{L_k} a_{l,k} e^{b_{l,k}t}, \qquad \operatorname{Re} b_{l,k} \leqslant 0.$$

In discretizing transparent boundary conditions this approach enables one to achieve an arbitrary accuracy by taking the appropriate number of Fourier harmonics and the corresponding approximations of $\mathrm{B}_k(t)$. On the other hand, it is economical in what concerns the memory resources and the number of operations, because convolution with exponentials is carried out using recurrence formulae. Transparent boundary conditions for some other equations were elaborated in [43]–[46] and other papers. For anisotropic and inhomogeneous media quasi-analytic transparent boundary conditions were developed in [47], [48], where the corresponding matrix $\{B_{k,l}^*\}$ is full and its matrix elements are found numerically. In addition, analytic truncated transparent boundary conditions have been proposed [49] in which there is no operator of time convolution.

Now we return to the problem of constructing a difference fundamental solution for a wave equation with constant coefficients. It could nevertheless be solved thanks to V. I. Turchaninov's investigations [50]. It turned out that once we have replaced the point-mass delta function on the right-hand side of the wave equation by some infinitely smooth 'cap' with compact support that simulates the delta function, the difference solution starts tending to zero outside the front of the fundamental solution, and at a much higher rate than prescribed by the second order of the difference scheme. In this way it became possible to use the presence of lacunas, something inherent to solutions of the wave equation, and to construct non-reflecting artificial boundary conditions on the outer open boundary of the computational domain which are virtually exact in the finite-difference sense, using fixed memory resources and a fixed number of operations in the computation at each time step [51]–[54]. These resources are proportional to the number of grid points in the computational domain, because the auxiliary domain introduced for the 'accommodation' of the non-reflecting artificial boundary conditions has a fixed diameter (approximately three times as big as the original computational domain).

Subsequently, this approach was significantly developed for various applications by Tsynkov and his colleagues [55]–[58] and was generalized to quasi-lacunas of

Maxwell's equations, when the solution behind the wave tail is stationary, but not necessarily zero [59], [60].

4) Ryaben'kii has always been very attentive to the efficiency of calculations. When multiprocessor systems appeared, the degree of parallelism of the computations became a most important characteristic of algorithms. The use of rectangular grids and finite differences in the original formulations of the theory of difference potentials makes possible a very efficient organization of high-performance computing. On the other hand, the standard approach to parallelism is to partition the physical and the computational domains. Ryaben'kii moved in this direction at the end of the 1990s, together with his student Y. Yu. Epshteyn and Turchaninov. Their first results confirmed the efficiency of their prospective approach to the construction of an algorithm for solving the problem using the method of difference potentials in a composite domain. They proposed algorithms [61], [62] which construct separate, not harmonized (rectangular) grids in different subdomains. The boundary and interface conditions are approximated directly on the physical boundaries themselves with the use of the operator described above which connects the Cauchy data with functions on the grid boundaries of the subdomains.

Another source of economy of computer memory and number of operations is improvement of the order of approximation of the difference scheme operators, and therefore also of the difference potential operators. In developing the method of difference potentials in this direction, Tsynkov and his colleagues have designed efficient algorithms for high-order approximations of solutions of the Helmholtz equation [63]–[66] on the basis of the compact schemes of the fourth order of accuracy previously proposed [67], [68]. We remark that these compact schemes for difference potentials are attractive also because they keep the boundary γ two-layered, in contrast to the four-layer grid boundary γ used for central-difference schemes of the fourth order. (However, Epshteyn and coauthors [69], [70] showed that in the design of algorithms with difference potentials for composite domains which are based on such central-difference schemes the additional layers of γ do not introduce fundamental difficulties.)

5) In the early 1990s Aleksei Valerievich Zabrodin, the head of a department at the Institute of Applied Mathematics and a close friend of Ryaben'kii who was one of the first enthusiasts in the development of Russian multiprocessor computer systems and parallel programming techniques, suggested that Ryaben'kii look at the problem of active shielding of physical fields. Ryaben'kii was immediately intrigued by this subject: he intuitively felt that difference potentials could be of great use there. In fact, the trace of a function on a grid boundary carries all the necessary and sufficient information for the recovery in the shielded region of just the field generated by the outside sources (and therefore the theory of difference potentials enables us to simulate this field and then to subtract it). Of course, in creating his algorithm Ryaben'kii's main care was the practical realization of the mathematical model by means of available physical gadgets and measurements. For instance, approaches involving the necessity of using information about the properties of the medium in the protected regions were immediately abandoned. Several months were spent obtaining the required formulae, and they turned out to be so simple that it was difficult to believe in the prospects they opened. Ryaben'kii published these

first theoretical results as two short notes [71] and [72]. They marked the beginning of a new direction in the well-known problem of active shielding of a given region of space from outside sources of noise, and they contained the main construction in the use of Cauchy-type difference potentials for these purposes.

For definiteness let us consider *acoustic fields.* By a 'sound' we will mean a useful field in the shielded region, and by a 'noise' we will mean a parasitic field which penetrates from an adjacent region across the boundary between the regions (one example is a room with a window looking out at a noisy street). We recall that the mathematical aspect of the problem of constructing an active screening system (a problem which has been studied for more than 50 years) consists mainly in describing a) a system of additional sound sources located on the boundary of the shielded region which dampen the noise penetrating from the open part of the common boundary with the outside region, that is, which protect silence in the given subregion, and b) a control of this system. The new problem considered by Ryaben'kii consisted in protecting from outside noise not silence but an arbitrary useful sound in the given subdomain.

We have already seen that the apparatus of difference potentials uses certain special grid sets. Let us introduce some notation: M is the shielded grid domain (the room), M^- is the adjacent grid domain (the street), and γ is the multilayer grid interface between these regions (the window opening). We assume that the acoustic field is described by finite-difference equations in these domains. The formulae for active control constructed in [71] and [72] on the basis of Cauchy-type difference potentials, applied to a *time-periodic* acoustic field, have some advantages over all the previously known mathematical models of active shielding systems, because they possess a combination of the following properties.

1. In M not only is the noise cancelled, but also the sound produced in this region is preserved. In other words, the acoustic field in M becomes the same as it is when the noise sources in M^- are shut off. Moreover, for the sound sources in M even the reflections (echoes) from obstacles in M^- transmitted through γ are preserved.

2. The proposed algorithm of active shielding operates with the values of the total acoustic field at points in γ that is produced by sources both in M^- and in M. Of course, this simplifies signal measuring and processing.

3. For use of the algorithm it is necessary to know the properties of the acoustic medium only in the immediate vicinity of the grid boundary γ. In particular, the shape of the composite region and the conditions on its outer boundary are irrelevant, and we do not need to know the location and strength of the sound and noise sources, nor the state of the medium away from the grid boundary.

The periodic dependence of the field on time treated by the theory in [71] and [72] can have a simple harmonic form (the Helmholtz equation) or can be produced by some repeating process. In the latter case we can proceed as follows: store in memory the values of the field at the points in γ during the first run of the process, calculate the required control, and then use it at each repetition.

The universal results of the approach proposed in the pioneering paper [71] have served as the basis for numerous investigations in this direction carried out by

Ryaben'kii and his colleagues R. I. Veitsman, E. V. Zinov'ev, Tsynkov, S. V. Utyuzh-nikov, and their students and colleagues, who have developed and used them for diverse applications (see, for instance, [73]–[83]).

In 2005 Ryaben'kii visited the University of Manchester at the invitation of Utyuzhnikov. As a result of his lectures at seminars and many conversations, he and Utyuzhnikov decided to build a laboratory facility for active noise control on the basis of the universal algorithm in [71], specialized for the case of one-dimensional acoustic difference equations. Experiments at the facility in Manchester constructed under Utyuzhnikov's leadership [84] have validated the theoretical predictions in [71].

For all the merits of active screening systems based on [71], one cannot use this algorithm to control *stochastic* acoustic processes in real time, that is, in the situation when at the current moment of time $t = T$ one does not have information about the variable shape of the region and the sources of noise and sound for $t \geqslant T$. We can also add here the following restrictions (which we formulate for the 'room-window-street' situation): a) the microphones and sound emitters must be placed in the 'window opening', that is, close to γ; b) the required control algorithm must use for $t = T$ only the information about the total acoustic field on the time interval $0 \leqslant t \leqslant T$.

Many years of speculations on this and similar problems of real-time control of stochastic processes led Ryaben'kii to the following conclusions.

1. The protected region M cannot be completely shielded from the noise from M^-, because the required control is underdetermined in view of the restrictions on the available information in the vicinity of γ.

2. Nevertheless, it is possible to reduce the noise in M by any prescribed factor n.

The corresponding theory and algorithm were presented in [85]–[89]. The gist of this algorithm is that the information not available for control about the acoustic conditions away from the window that change with time (moving objects, temperature, rain, snow, sound sources, and so on) arrives to the window opening as weak noise (for large n) which is preserved because of the abandonment of the goal of total noise cancellation. In essence, this implements the idea of location by means of weak noise, which is detected by microphones and used for the timely production of the current control signal.

The next step in the development of the prospective laboratory facility for real-time active reduction of stochastic noise was a mathematical model developed by Ryaben'kii and Turchaninov that realized the above algorithm as computer software. In this model the acoustic process in a continuous medium is calculated using a stable finite-difference scheme which approximates the acoustic equations on a sufficiently fine grid. The first numerical experiments validated the theoretical predictions [85]–[89] and established some new facts [90].

It should be noted that there is a very natural connection between the discrete and continuous formulations of the problem of active shielding, due to the similar algebraic formalisms of the potentials in (2) and (5) and to the approximative properties of difference potentials considered above.

Completing our brief description of applications of difference potentials with this striking example, we are quite sure that the method of difference potentials developed by Ryaben'kii more than 45 years ago will produce many other interesting

and surprising solutions in various important areas of numerical mathematics and engineering in the future.

Ryaben'kii has a rare ability to inspire students and younger colleagues with his ideas by fascinating them with accounts of the possibilities and unsolved problems involving difference potentials. He has no regrets about time spent talking with them, and he generously shares his extremely rich scientific experience and life experience with them. He addresses the question of the future line of research of a young colleague with the utmost sense of responsibility and forms around himself an inimitable atmosphere of enthusiasm and creativity which enables future researchers to fully realize their capabilities. For Ryaben'kii and his late wife Natal'ya Petrovna Ryaben'kaya (she passed away on 15 January 2014), who lived for their common interests, each student became very close to them. This is one reason why Ryaben'kii has not had as many students as he could have had (another reason was that in the times of 'perestroika' many of his talented students simply abandoned science). About ten of his students defended Ph.D. dissertations, and two of them were subsequently awarded D.Sc. degrees.

The school of research founded by Ryaben'kii currently consists of his students and students of his students, working in Russia and abroad, in Great Britain, Germany, Israel, and the USA. The ideas and methods that he has put forward have been developed significantly in theoretical and applied research studies at universities, government laboratories (NASA, DOD), and research centres of major industrial corporations (Rosneft, Schlumberger, ALSTROM, EDF).

Of course, Ryaben'kii does not work with students any more, nor does he teach, but rather expends his strengths only concerning those of his ideas where his experience and knowledge are needed for accomplishing specific results. Although he began teaching very early, after completing his postgraduate studies in the 1950s, his talents as a teacher and lecturer revealed themselves only later, during the 30 years of his work in the Department of Numerical Mathematics at the Moscow Institute for Physics and Technology. He created a completely original lecture course on the foundations and methods of computational mathematics, many chapters in which are unique. His work on these lectures and his monograph [4] led to the issue of the coursebook [5], which has now become a standard textbook in the corresponding branch of computational mathematics and has been translated into many languages. The coursebooks [91] and [92] (joint with Tsynkov) sum up the years of his teaching experience. His long involvement in seminars is also reflected in the book of practical exercises on basic computational mathematics, including PC software [93], written by the staff and students of Ryaben'kii's department under his supervision. The theory of difference potentials was presented in the monographs [18] and [94], which also contain many results (revised in each new edition: 1987, 2002, 2010) concerning applications of difference potentials to problems in mathematical physics. In total, Ryaben'kii has published about a dozen textbooks and monographs and more than 140 research papers in Russian and international journals.

Ryaben'kii is a professor of the Moscow Institute for Physics and Technology (since 1970) and a principal researcher in the Keldysh Institute of Applied Mathematics of the Russian Academy of Sciences. He is an Honoured Science Worker of

the Russian Federation (2005) and a winner of the I. G. Petrovsky Prize of the Presidium of the Russian Academy of Sciences (2007) for his book [94]. Several research seminars supervised by Ryaben'kii have worked actively for many years during different periods of time. In 1998 a special section was organized in the framework of the international conference ICOSAHOM'98 to honour the 50 years of Ryaben'kii's research activity and his outstanding contributions to computational mathematics. In 2013 the international conference in Moscow "Difference Schemes and Their Applications" was dedicated to his 90th birthday. The talks at this conference filled a special issue of the journal *Applied Numerical Mathematics* [95].

In summarizing briefly the contributions to applied mathematics already made by Ryaben'kii, we can list the following concepts firmly associated in our minds with his name: the Ryaben'kii–Filippov 'convergence theorem', the Godunov–Ryaben'kii 'spectrum of a family of difference operators', Ryaben'kii's 'smooth local interpolation', Ryaben'kii's 'difference potentials', and Ryaben'kii's 'algorithm for active noise reduction'.

At the age of 92, Ryaben'kii lives still fascinated with his favorite subject, mathematics. He is always cheerful and optimistic, and has an amazingly clear mind. All who come in contact with him are immediately attracted by his sincerity, firmness of principles, and kindness. Anyone fortunate enough to spend some time with Ryaben'kii in friendly conversation or at meetings in the Institute of Applied Mathematics or elsewhere will forever remember his recollections of episodes of war, of his friends and colleagues, so wholehearted and emotional, full of vivid characterizations as they are. We maintain the most kind and grateful sentiments towards Viktor Solomonovich and wish our dear friend, colleague, and teacher new successes in his creative work, a long physical and scientific life, good health, and happiness.

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