A high order method of boundary operators for the 3D time-dependent wave equation

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Abstract

We propose an efficient high order accurate boundary algorithm for the numerical solution of unsteady exterior initial boundary problems for the three-dimensional wave equation. The algorithm relies on the method of difference potentials combined with the Huygens' principle.

Keywords: method of difference potentials, Huygens' principle, unsteady wave propagation

1 Introduction

Consider an exterior initial boundary value problem (IBVP) for the three-dimensional homogeneous wave (d'Alembert) equation:

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \Delta u = 0, \quad \text{on} \quad \mathbb{R}^3\backslash\Omega \times [0,T], \tag{1a}$$

$$\boldsymbol{l}_\Gamma u = \phi, \quad \text{on} \quad \partial\Omega \times [0,T], \tag{1b}$$

$$u|_{t=0} = \partial u/\partial t|_{t=0} = 0, \tag{1c}$$

where c is the speed of light. The boundary condition (1b) is inhomogeneous. For example, if u is the field scattered off the given shape Ω, then the operator $\boldsymbol{l}_\Gamma$ defines the type of scattering on $\partial\Omega$ and the data ϕ represent the impinging field.

The numerical method we propose for solving the IBVP (1) combines the flexibility and ease of finite differences with the advantages of a boundary approach. It reduces the dimension of the problem by one and handles non-conforming boundaries $\partial\Omega$ on regular grids with no loss of accuracy. These features are enabled by the method of difference potentials (MDP) [1] that employs discrete counterparts of Calderon's operators. The MDP has previously been used for the simulation of both time-harmonic [2] and time-dependent waves [3]. As an extension of [3], the current work addresses exterior problems and offers high order accuracy.

A fundamental difficulty in applying boundary methods (e.g., those based on retarded potential boundary integral equations [4] or those based on Calderon's operators) to time-dependent problems is that the boundary extends with time. To avoid the growth of cost, we employ the strong Huygens' principle that helps us truncate the ever expanding "tail" of the algorithm. It also guarantees that only outgoing waves will be present in the solution to the exterior problem. The time marching is therefore performed on a sliding window of fixed duration and only along the boundary $\partial\Omega \times [0,T]$, which has dimension (2+1) in space-time. As such, the method provides sub-linear complexity, i.e., outperforms the typical explicit schemes in long-time simulations. Moreover, changing the boundary condition (1b) incurs only a minor additional cost compared to that for a conventional volumetric time-stepping technique.

2 Method

The MDP reduces the PDE (1a) from the unbounded domain $\mathbb{R}^3\backslash\Omega \times [0,T]$ to the operator equation at the boundary $\Gamma = \partial\Omega \times [0,T]$:

$$\boldsymbol{P}_\Gamma \boldsymbol{\xi}_\Gamma = \boldsymbol{\xi}_\Gamma. \tag{2}$$

In (2), $\boldsymbol{P}_\Gamma$ is Calderon's projection for the d'Alembert operator and $\boldsymbol{\xi}_\Gamma \equiv (\xi_0, \xi_1)$ is the density of the generalized Calderon's potential:

$$\boldsymbol{P}_\Omega \boldsymbol{\xi}_\Gamma = \int_\Gamma \Big\{ \xi_1(\boldsymbol{y},t') G(\boldsymbol{x}-\boldsymbol{y}, t-t') - \xi_0(\boldsymbol{y},t') \frac{\partial G}{\partial \boldsymbol{n}}(\boldsymbol{x}-\boldsymbol{y}, t-t') \Big\} dt' dS_{\boldsymbol{y}}. \tag{3}$$

The functions ξ_0 and ξ_1 in (2), (3) are traces of the solution and its normal derivative on Γ, respectively, and $G(\boldsymbol{x},t)$ in (3) is the fundamental solution of the d'Alembert operator. The projection $\boldsymbol{P}_\Gamma$ in (2) is the vector trace of the potential (3): $\boldsymbol{P}_\Gamma \boldsymbol{\xi}_\Gamma = \big(\boldsymbol{P}_\Omega \boldsymbol{\xi}_\Gamma, \frac{\partial \boldsymbol{P}_\Omega \boldsymbol{\xi}_\Gamma}{\partial \boldsymbol{n}}\big)\big|_\Gamma$.

The boundary equation (2), which is equivalent to (1a), is solved as a system along with the BC (1b). This can be arbitrary as long as the overall formulation (1) is well-posed. In simple cases, the BC explicitly provides one component of $\boldsymbol{\xi}_\Gamma$, e.g., ξ_0 for a Dirichlet BC and ξ_1

for a Neumann BC. The remaining component is then obtained as a solution to (2).

To discretize (2), the MDP computes the finite difference projection operator by solving a series of inhomogeneous auxiliary problems (APs) for equation (1a). The AP is originally formulated as a Cauchy problem and then truncated to a bounded domain Ω_0 of simple shape, see Figure 1, where it can be easily integrated by any appropriate finite difference scheme. The Huygens' principle combined with MDP enables a perfectly reflectionless treatment of the artificial outer boundary $\partial\Omega_0$, as in [5].

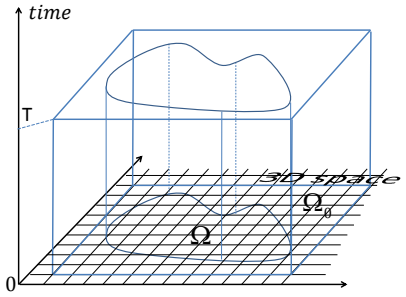

Figure 1: Computational domain for the AP.

Moreover, the Huygens' principle incorporated into our time-marching algorithm implies that for a bounded domain Ω in space, the extent of the backward dependence of the solution u to equation (1a) in time is finite and non-increasing. This property allows us to solve (2) (and thus, (1)) over long computational times $T_{\text{final}} \gg T$ sequentially, updating the density $\boldsymbol{\xi}_\Gamma$ by "chunks" of size T, see Figure 2. The solution u on Ω is computed only once, at $t = T_{\text{final}}$.

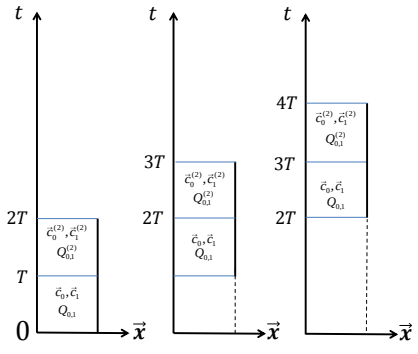

Figure 2: Time marching by "chunks" of size T.

3 Numerical demonstrations

To solve the AP, we employed a fourth order accurate compact scheme [6] (it controls the dispersion error more efficiently than the previously used lower order schemes). The test problem is scattering of a plane wave about a sphere of radius R_0. All computations are conducted on a Cartesian grid, for which the spherical boundary $r = R_0$ is non-conforming. Figure 3 shows the error profiles for a long-time run with $T_{\text{final}} = 4000 R_0/c$ on two consecutive grids. Table 1 demonstrates that the CPU time to advance the solution over $T = R_0/c$ scales roughly as $2^3 = 8$ for the proposed boundary method versus $2^4 = 16$ for an explicit volumetric scheme.

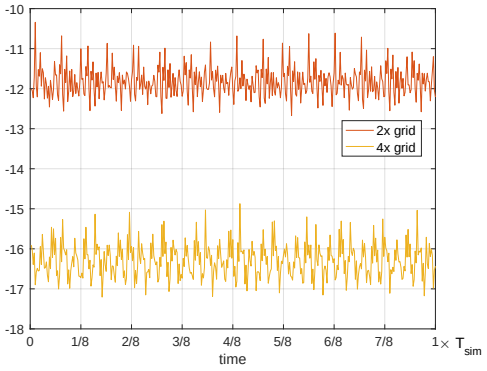

Figure 3: Fourth order convergence of the proposed boundary method with Robin BC (1b).

Grid	CPU time, sec		scaling rate	
	MDP	Volume	MDP	Volume
1×	0.0474	1.26	-	-
2×	0.421	19.8	8.87	15.7
4×	3.56	322	8.46	16.3

Table 1: Comparison of numerical performance over a fixed time interval $T = R_0/c$ of the proposed boundary method (MDP+Huygens' principle) vs. standard volumetric time marching.

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