10.3 Spectral Stability Criterion for Finite-Difference Cauchy Problems

Perhaps the most widely used approach to the analysis of stability for finite-difference Cauchy problems has been proposed by von Neumann. In this section, we will introduce and illustrate it using several examples of difference equations with constant coefficients. The case of variable coefficients will be addressed in Section 10.4 and the case of initial boundary value problems (as opposed to pure initial value problems) will be explored in Section 10.5.

10.3.1 Stability with Respect to Initial Data

The simplest example of a finite-difference Cauchy problem is the first order up-wind scheme:

$$\boldsymbol{L}_h u^{(h)} = f^{(h)}, \tag{10.69}$$

for which the operator $\boldsymbol{L}_h$ and the right-hand side $f^{(h)}$ are given by:

$$\begin{aligned}
\boldsymbol{L}_h u^{(h)} &= \begin{cases} \dfrac{u_m^{p+1} - u_m^p}{\tau} - \dfrac{u_{m+1}^p - u_m^p}{h}, & m = 0, \pm 1, \pm 2, \dots, \\ & p = 0, 1, \dots, [T/\tau] - 1, \\ u_m^0, & m = 0, \pm 1, \pm 2, \dots, \end{cases} \\
f^{(h)} &= \begin{cases} \varphi_m^p, \quad m = 0, \pm 1, \pm 2, \dots, & p = 0, 1, \dots, [T/\tau] - 1, \\ \psi_m, \quad m = 0, \pm 1, \pm 2, \dots. \end{cases}
\end{aligned} \tag{10.70}$$

We have already encountered this scheme on many occasions. Let us define the norms $\|u^{(h)}\|_{U_h}$ and $\|f^{(h)}\|_{F_h}$ as maximum norms (alternatively called l_∞ or C):

$$\|u^{(h)}\|_{U_h} \stackrel{\text{def}}{=} \max_p \sup_m |u_m^p|, \quad \|f^{(h)}\|_{F_h} \stackrel{\text{def}}{=} \max_p \sup_m |\varphi_m^p| + \sup_m |\psi_m|. \tag{10.71}$$

Then for the scheme (10.69)–(10.70), the stability condition (10.17):

$$\|u^{(h)}\|_{U_h} \leq c \|f^{(h)}\|_{F_h}$$

given by Definition 10.2 transforms into:

$$\sup_m |u_m^p| \leq c \left[\max_p \sup_m |\varphi_m^p| + \sup_m |\psi_m| \right], \quad p = 0, 1, \dots, [T/\tau], \tag{10.72}$$

where the constant c is not supposed to depend on h (or on $\tau = rh$, $r = \text{const}$).

Condition (10.72) must hold for any arbitrary $\{\psi_m\}$ and $\{\varphi_m^p\}$. In particular, it should obviously hold for an arbitrary $\{\psi_m\}$ and $\{\varphi_m^p\} \equiv 0$. In other words, for stability it is necessary that solution u_m^p to the problem:

$$\frac{u_m^{p+1}-u_m^p}{\tau}-\frac{u_{m+1}^p-u_m^p}{h}=0,\quad m=0,\pm1,\pm2,\dots,\quad p=0,1,\dots,[T/\tau]-1,$$
$$u_m^0=\psi_m,\quad m=0,\pm1,\pm2,\dots,\tag{10.73}$$

satisfy the inequality:

$$\sup_m|u_m^p|\le c\sup_m|\psi_m|,\quad p=0,1,\dots,[T/\tau],\tag{10.74}$$

for any bounded grid function ψ_m.

Property (10.74), which is *necessary* for the finite-difference scheme (10.69)–(10.70) to be stable in the sense (10.72), is called *stability with respect to perturbations of the initial data,* or simply stability with respect to the initial data. It means that if a perturbation is introduced into the initial data ψ_m of problem (10.73), then the corresponding perturbation of the solution will be no more than c times greater in magnitude than the original perturbation of the data, where the constant c does not depend on the grid size h.

10.3.2 A Necessary Spectral Condition for Stability

For the Cauchy problem (10.69)–(10.70) to be stable with respect to the initial data it is necessary, in particular, that inequality (10.74) hold for ψ_m being equal to a harmonic function:

$$u_m^0=\psi_m=e^{i\alpha m},\quad m=0,\pm1,\pm2,\dots,\tag{10.75}$$

where α is a real parameter. One can easily solve problem (10.73) with the initial condition (10.75); the solution u_m^p can be found in the form:

$$u_m^p=\lambda^p e^{i\alpha m},\quad m=0,\pm1,\pm2,\dots,\quad p=0,1,\dots,[T/\tau],\tag{10.76}$$

where the quantity $\lambda=\lambda(\alpha)$ is determined by substitution of expression (10.76) into the homogeneous finite-difference equation of problem (10.73):

$$\lambda(\alpha)=1-r+re^{i\alpha},\quad r=\tau/h=\text{const}.\tag{10.77}$$

Solution (10.76) satisfies the equality:

$$\sup_m|u_m^p|=|\lambda(\alpha)|^p\sup_m|u_m^0|=|\lambda(\alpha)|^p\sup_m|\psi_m|.$$

Therefore, for the stability condition (10.74) to be true it is necessary that the following inequality hold for all real α:

$$|\lambda(\alpha)|^p\le c,\quad p=0,1,\dots,[T/\tau].$$

Equivalently, we can require that

$$|\lambda(\alpha)| \leq 1 + c_1\tau, \tag{10.78}$$

where c_1 is a constant that does not depend either on α or on τ. Inequality (10.78) represents the necessary spectral condition for stability due to von Neumann. It is called spectral because of the following reason.

Existence of the solution in the form (10.76) shows that the harmonic $e^{i\alpha m}$ is an eigenfunction of the transition operator from time level t_p to time level t_{p+1}:

$$u_m^{p+1} = (1-r)u_m^p + ru_{m+1}^p, \quad m = 0, \pm 1, \pm 2, \ldots.$$

According to the finite-difference equation (10.73), this operator maps the grid function $\{u_m^p\}$, $m = 0, \pm 1, \pm 2, \ldots$, defined for $t = t_p$ onto the grid function $\{u_m^{p+1}\}$, $m = 0, \pm 1, \pm 2, \ldots$, defined for $t = t_{p+1}$. The quantity $\lambda(\alpha)$ given by formula (10.77) is therefore an eigenvalue of the transition operator that corresponds to the eigenfunction $\{e^{i\alpha m}\}$. In the literature, $\lambda(\alpha)$ is sometimes also referred to as the amplification factor, we have encountered this concept in Section 10.2.3. The set of all complex numbers $\lambda = \lambda(\alpha)$ obtained when the parameter α sweeps through the real axis forms a curve on the complex plane. This curve is called the spectrum of the transition operator.

Consequently, the necessary stability condition (10.78) can be re-formulated as follows: The spectrum of the transition operator that corresponds to the difference equation of problem (10.73) must belong to the disk of radius $1 + c_1\tau$ centered at the origin on the complex plane. In our particular example, the spectrum (10.77) does not depend on τ at all. Therefore, condition (10.78) is equivalent to the requirement that the spectrum $\lambda = \lambda(\alpha)$ belong to the unit disk:

$$|\lambda(\alpha)| \leq 1. \tag{10.79}$$

Let us now use the criterion that we have formulated, and actually analyze stability of problem (10.69)–(10.70). The spectrum (10.77) forms a circle of radius r centered at the point $(1-r, 0)$ on the complex plane. When $r < 1$, this circle lies inside the unit disk, being tangent to the unit circle at the point $\lambda = 1$. When $r = 1$ the spectrum coincides with the unit circle. Lastly, when $r > 1$ the spectrum lies outside the unit disk, except one point $\lambda = 1$, see Figure 10.5. Accordingly, the necessary stability condition (10.79) is satisfied for $r \leq 1$ and is violated for $r > 1$. Let us now recall that in Section 10.1.3 we have studied the same difference problem and have proven that it is stable when $r \leq 1$ and is unstable when $r > 1$. Therefore, in this particular case the von Neumann necessary stability condition appears sufficiently sensitive to distinguish between the actual stability and instability.

In the case of general Cauchy problems for finite-difference equations and systems, we give the following

DEFINITION 10.3 *The spectrum of a finite-difference problem is given by the set of all those and only those $\lambda = \lambda(\alpha, h)$, for which the corresponding*

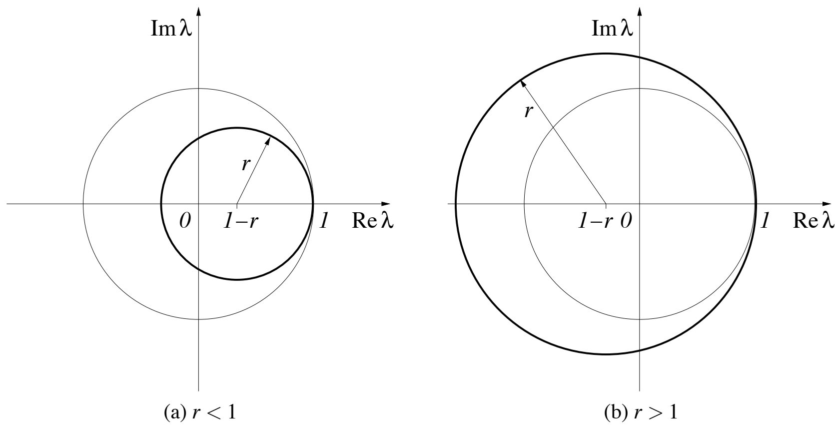

FIGURE 10.5: Spectra of the transition operators for the upwind scheme.

homogeneous finite-difference equation or system has a solution of the form:

$$\boldsymbol{u}_m^p = \lambda^p \left[\boldsymbol{u}^0 e^{i\alpha m}\right], \quad m = 0, \pm 1, \pm 2, \ldots, \quad p = 0, 1, \ldots, [T/\tau], \tag{10.80}$$

where $\boldsymbol{u}^0$ is either a fixed number in the case of one scalar difference equation (this number may be taken equal to one with no loss of generality), or a constant finite-dimensional vector in the case of a system of difference equations.

Then, the von Neumann necessary stability condition says that given an arbitrarily small number $\varepsilon > 0$, for all sufficiently small grid sizes h the spectrum $\lambda = \lambda(\alpha, h)$ of the difference problem has to lie inside the following disk on the complex plane:

$$|\lambda| \leq 1 + \varepsilon. \tag{10.81}$$

Note that if for a particular problem the spectrum appears to be independent of the grid size h (and τ), then condition (10.81) becomes equivalent to the requirement that the spectrum $\lambda(\alpha, h) = \lambda(\alpha)$ belong to the unit disk, see (10.79).

If the von Neumann necessary condition (10.81) is not satisfied, then one should not expect stability for any reasonable choice of norms. If, on the other hand, this condition is met, then one may hope that for a certain appropriate choice of norms the scheme will turn out stable.

10.3.3 Examples

We will exploit the von Neumann spectral condition to analyze stability of a number of interesting finite-difference problems. First, we will consider the schemes that approximate the Cauchy problem:

$$\begin{aligned} \frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} &= \varphi(x,t), \quad -\infty < x < \infty, \quad 0 < t \leq T, \\ u(x,0) &= \psi(x), \quad -\infty < x < \infty. \end{aligned} \tag{10.82}$$

Example 1

Consider the first order downwind scheme:

$$\frac{u_m^{p+1}-u_m^p}{\tau}-\frac{u_m^p-u_{m-1}^p}{h}=\varphi_m^p, \quad m=0,\pm1,\pm2,\dots,\ p=0,1,\dots,[T/\tau]-1,$$
$$u_m^0=\psi_m, \quad m=0,\pm1,\pm2,\dots.$$

Substituting the solution of type (10.76) into the corresponding homogeneous finite-difference equation, we obtain:

$$\lambda(\alpha)=1+r-re^{-i\alpha}.$$

Therefore, the spectrum is a circle of radius r centered at the point $(1+r,0)$ on the complex plane, see Figure 10.6. This spectrum does not depend on h. It is also clear that for no value of r does it belong to the unit circle. Consequently, the stability condition (10.79) may never be satisfied. This conclusion is expected, because the downwind scheme obviously violates the Courant, Friedrichs, and Lewy condition for any $r=\tau/h$ (see Section 10.1.4).

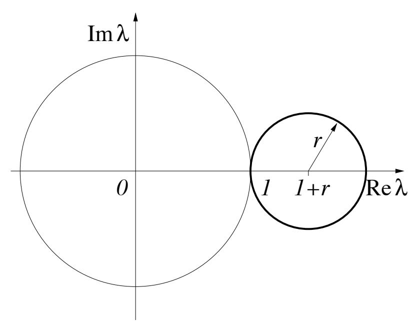

FIGURE 10.6: Spectrum for the downwind scheme.

Example 2

Next, consider the Lax-Wendroff scheme:

$$\frac{u_m^{p+1}-u_m^p}{\tau}-\frac{u_{m+1}^p-u_{m-1}^p}{2h}-\frac{\tau}{2h^2}(u_{m+1}^p-2u_m^p+u_{m-1}^p)=\varphi_m^p, \qquad (10.83)$$
$$u_m^0=\psi_m,$$

that approximates problem (10.82) with second order accuracy if $\varphi\equiv0$, and with first order accuracy otherwise. For this scheme, the spectrum $\lambda=\lambda(\alpha,h)$ is determined from the equation:

$$\frac{\lambda-1}{\tau}-\frac{e^{i\alpha}-e^{-i\alpha}}{2h}-\frac{\tau}{2h^2}(e^{i\alpha}-2+e^{-i\alpha})=0.$$

Let us denote $r=\tau/h$ as before, and notice that

$$\frac{e^{i\alpha}-e^{-i\alpha}}{2i}=\sin\alpha,$$
$$\frac{e^{i\alpha}-2+e^{-i\alpha}}{4}=-\left(\frac{e^{i\alpha/2}-e^{-i\alpha/2}}{2i}\right)^2=-\sin^2\frac{\alpha}{2}.$$

Then,

$$\lambda(\alpha) = 1 + ir\sin\alpha - 2r^2\sin^2\frac{\alpha}{2},$$
$$|\lambda(\alpha)|^2 = \left(1 - 2r^2\sin^2\frac{\alpha}{2}\right)^2 + r^2\sin^2\alpha.$$

The spectrum does not depend on h, and from the previous equality we easily find that

$$1 - |\lambda|^2 = 4r^2(1-r^2)\sin^4\frac{\alpha}{2}.$$

The von Neumann condition is satisfied when the right-hand side of this equality is non-negative, which means $r \leq 1$; it is violated when $r > 1$.

Example 3

Consider the explicit central-difference scheme:

$$\frac{u_m^{p+1} - u_m^p}{\tau} - \frac{u_{m+1}^p - u_{m-1}^p}{2h} = \varphi_m^p, \quad m = 0, \pm 1, \pm 2, \ldots, \; p = 0, 1, \ldots, [T/\tau] - 1,$$
$$u_m^0 = \psi_m, \quad m = 0, \pm 1, \pm 2, \ldots, \tag{10.84}$$

for the same Cauchy problem (10.82).

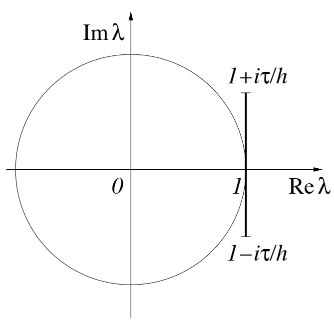

FIGURE 10.7: Spectrum for scheme (10.84).

Substituting expression (10.76) into the homogeneous counterpart of the difference equation from (10.84), we have:

$$\frac{\lambda - 1}{\tau} - \frac{e^{i\alpha} - e^{-i\alpha}}{2h} = 0,$$

which yields:

$$\lambda(\alpha) = 1 + i\left(\frac{\tau}{h}\sin\alpha\right).$$

The spectrum $\lambda = \lambda(\alpha)$ fills the vertical interval of length $2\tau/h$ that crosses through the point $(1,0)$ on the complex plane, see Figure 10.7.

If $\tau/h = r = \text{const}$, then the spectrum can be said to be independent of h (and of τ). This spectrum lies outside of the unit disk, the von Neumann condition (10.79) is not met, and the scheme is unstable. However, if we require that the temporal size τ be proportional to h^2 as $h \longrightarrow 0$, so that $r = \tau/h^2 = \text{const}$, then the point of the spectrum, which is most distant from the origin, will be $\lambda(\pi/2)$ [and also $\lambda(-\pi/2)$]. For this point we have:

$$|\lambda(\pi/2)| = \sqrt{1 + \left(\frac{\tau}{h}\right)^2} = \sqrt{1 + r\tau} < 1 + \frac{r}{2}\tau.$$

Then, the von Neumann condition (10.81) in the form $\lambda(\alpha) \leq 1 + c_1\tau$ is satisfied for $c_1 = r/2$. It is clear that the requirement $\tau = rh^2$ puts a considerably stricter constraint on the rate of decay of the temporal grid size τ as $h \longrightarrow 0$ than the previous requirement $\tau = rh$ does. Still, that previous requirement was sufficient for the von Neumann condition to hold for the difference schemes (10.69)–(10.70) and (10.83) that approximate the same Cauchy problem (10.82).

We also note that the Courant, Friedrichs, and Lewy condition of Section 10.1.4 allows us to claim that the scheme (10.84) is unstable only for $\tau/h = r > 1$, but does not allow us to judge the stability for $\tau/h = r \leq 1$. As such, it appears weaker than the von Neumann condition.

Example 4

The instability of scheme (10.84) for $\tau/h = r = \text{const}$ can be fixed by changing the way the time derivative is approximated. Instead of (10.84), consider the scheme:

$$\begin{aligned} \frac{u_m^{p+1} - (u_{m-1}^p + u_{m+1}^p)/2}{\tau} - \frac{u_{m+1}^p - u_{m-1}^p}{2h} &= \varphi_m^p, \\ u_m^0 &= \psi_m, \end{aligned} \tag{10.85}$$

obtained by replacing u_m^p with $(u_{m-1}^p + u_{m+1}^p)/2$. This is, in fact, a general approach attributed to Friedrichs, and the scheme (10.85) is known as the Lax-Friedrichs scheme; we have first introduced it in Section 10.2.4. The equation to determine the spectrum for the scheme (10.85) reads:

$$\frac{\lambda - (e^{i\alpha} + e^{-i\alpha})/2}{\tau} - \frac{e^{i\alpha} - e^{-i\alpha}}{2h} = 0,$$

which yields:

$$\frac{\lambda - \cos\alpha}{\tau} - \frac{i\sin\alpha}{h} = 0$$

and

$$\lambda(\alpha) = \cos\alpha + ir\sin\alpha,$$

where $r = \tau/h = \text{const}$. Consequently,

$$|\lambda(\alpha)|^2 = \cos^2\alpha + r^2\sin^2\alpha.$$

Clearly, the von Neumann condition (10.79) is satisfied for $r \leq 1$, because then $|\lambda|^2 \leq \cos^2\alpha + \sin^2\alpha = 1$. For $r > 1$, the von Neumann condition is violated.

Example 5

Finally, consider the leap-frog scheme (10.36) for problem (10.82). The corresponding homogeneous finite-difference equation is written as:

$$\frac{u_m^{p+1} - u_m^{p-1}}{2\tau} - \frac{u_{m+1}^p - u_{m-1}^p}{2h} = 0, \tag{10.86}$$

and for the spectrum we obtain:

$$\frac{\lambda - \lambda^{-1}}{2\tau} - \frac{e^{i\alpha} - e^{-i\alpha}}{2h} = 0.$$

This is a quadratic equation with respect to λ:

$$\lambda^2 - i2r\lambda \sin\alpha - 1 = 0,$$

where $r = \tau/h = \text{const}$. The roots of this equation are given by:

$$\lambda_{1,2} = ir\sin\alpha \pm \sqrt{1 - r^2 \sin^2\alpha}.$$

We notice that when $r \le 1$, then $|\lambda_{1,2}|^2 = r^2\sin^2\alpha + (1 - r^2\sin^2\alpha) = 1$, so that the entire spectrum lies precisely on the unit circle and the von Neumann condition (10.79) is satisfied. Otherwise, when $r > 1$, we again take $\alpha = \pi/2$ and obtain: $|\lambda_1| = |ir + i\sqrt{r^2-1}| = r + \sqrt{r^2-1} > 1$, which means that the von Neumann condition is not met. We thus see that the von Neumann criterion can be applied to a finite-difference equation that connects more than two consecutive time levels on the grid. However, for $r = 1$ the two-step scheme (10.86) proves unstable, see Section 10.3.6.

Next, we will consider two schemes that approximate the following Cauchy problem for the heat equation:

$$\begin{aligned} \frac{\partial u}{\partial t} - a^2\frac{\partial^2 u}{\partial x^2} &= \varphi(x,t), \qquad -\infty < x < \infty,\ \ 0 < t \le T, \\ u(x,0) &= \psi(x), \qquad -\infty < x < \infty. \end{aligned} \tag{10.87}$$

Example 6

The first scheme is explicit:

$$\begin{gathered} \frac{u_m^{p+1} - u_m^p}{\tau} - a^2\frac{u_{m+1}^p - 2u_m^p + u_{m-1}^p}{h^2} = \varphi_m^p, \\ u_m^0 = \psi_m, \quad m = 0, \pm 1, \pm 2, \dots, \quad p = 0, 1, \dots, [T/\tau] - 1. \end{gathered}$$

It allows us to compute $\{u_m^{p+1}\}$ in the closed form via $\{u_m^p\}$:

$$u_m^{p+1} = (1 - 2ra^2)u_m^p + ra^2(u_{m+1}^p + u_{m-1}^p) + \tau\varphi_m^p, \quad p = 0, 1, \dots, [T/\tau] - 1,$$

where $r = \tau/h^2 = \text{const}$. Substitution of $u_m^p = \lambda^p e^{i\alpha m}$ into the corresponding homogeneous difference equation yields:

$$\frac{\lambda - 1}{\tau} - a^2\frac{e^{-i\alpha} - 2 + e^{i\alpha}}{h^2} = 0.$$

By noticing that

$$\frac{e^{-i\alpha} - 2 + e^{i\alpha}}{4} = -\sin^2\frac{\alpha}{2},$$

we obtain:

$$\lambda(\alpha) = 1 - 4ra^2 \sin^2 \frac{\alpha}{2}, \quad r = \frac{\tau}{h^2}.$$

When α varies between 0 and 2π, the point $\lambda(\alpha)$ sweeps the interval $1 - 4ra^2 \leq \lambda \leq 1$ of the real axis, see Figure 10.8.

For stability, it is necessary that the left endpoint of this interval still be inside the unit circle (Figure 10.8), i.e., that $1 - 4ra^2 \geq -1$. This requirement translates into:

$$r \leq \frac{1}{2a^2}. \qquad (10.88)$$

Inequality (10.88) guarantees that the von Neumann stability condition will hold. Conversely, if we have $r > 1/(2a^2)$, then the point $\lambda(\alpha) = 1 - 4ra^2 \sin^2(\alpha/2)$ that corresponds to $\alpha = \pi$ will be located to the left of the point -1, i.e., outside the unit circle. In this case, the harmonic $e^{i\pi m} = (-1)^m$ generates the solution:

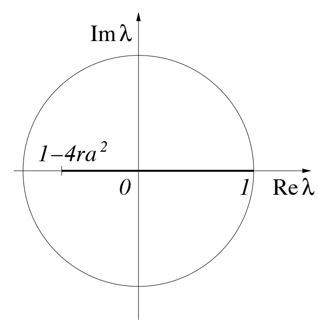

FIGURE 10.8: Spectrum of the explicit scheme for the heat equation.

$$u_m^p = (1 - 4ra^2)^p (-1)^m$$

that does not satisfy inequality (10.74) for any value of c.

Example 7

The second scheme is implicit:

$$\frac{u_m^{p+1} - u_m^p}{\tau} - a^2 \frac{u_{m+1}^{p+1} - 2u_m^{p+1} + u_{m-1}^{p+1}}{h^2} = \varphi_m^{p+1},$$
$$u_m^0 = \psi_m, \quad m = 0, \pm 1, \pm 2, \ldots, \quad p = 0, 1, \ldots, [T/\tau] - 1.$$

For this scheme, $\{u_m^{p+1}\}$ cannot be obtained from $\{u_m^p\}$ by an explicit formula because for a given m the difference equation contains three, rather than one, unknowns: u_{m+1}^{p+1}, u_m^{p+1}, and u_{m-1}^{p+1}. As in the previous case, we substitute $u_m^p = \lambda^p e^{i\alpha m}$ into the homogeneous equation and obtain:

$$\lambda(\alpha) = \frac{1}{1 + 4ra^2 \sin^2(\alpha/2)}, \quad r = \frac{\tau}{h^2}.$$

Consequently, the spectrum of the scheme fills the interval:

$$\left(1 + 4ra^2\right)^{-1} \leq \lambda \leq 1$$

of the real axis and the von Neumann condition $|\lambda| \leq 1$ is met for any r.

Example 8

The von Neumann analysis also applies when studying stability of a scheme in the case of more than one spatial variable. Consider, for instance, a Cauchy problem for the heat equation on the (x,y) plane:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \quad -\infty < x,\, y < \infty,\ \ 0 < t \leq T,$$
$$u(x,y,0) = \psi(x,y), \quad -\infty < x,\, y < \infty.$$

We approximate this problem on the uniform Cartesian grid: $(x_m, y_n, t_p) = (mh, nh, p\tau)$. Replacing the derivatives with difference quotients we obtain:

$$\frac{u_{m,n}^{p+1} - u_{m,n}^{p}}{\tau} = \frac{u_{m+1,n}^{p} - 2u_{m,n}^{p} + u_{m-1,n}^{p}}{h^2} + \frac{u_{m,n+1}^{p} - 2u_{m,n}^{p} + u_{m,n-1}^{p}}{h^2}$$
$$u_{m,n}^{0} = \psi_{m,n}, \quad m,n = 0, \pm 1, \pm 2, \ldots, \quad p = 0, 1, \ldots, [T/\tau] - 1.$$

The resulting scheme is explicit. To analyze its stability, we specify $u_{m,n}^{0}$ in the form of a two-dimensional harmonic $e^{i(\alpha m + \beta n)}$ determined by two real parameters α and β, and generate a solution in the form:

$$u_{m,n}^{p} = \lambda^{p}(\alpha,\beta) e^{i(\alpha m + \beta n)}.$$

Substituting this expression into the homogeneous difference equation, we find after some easy equivalence transformations:

$$\lambda(\alpha,\beta) = 1 - 4r\left(\sin^2\frac{\alpha}{2} + \sin^2\frac{\beta}{2}\right).$$

When the real quantities α and β vary between 0 and 2π, the point $\lambda = \lambda(\alpha,\beta)$ sweeps the interval $1 - 8r \leq \lambda \leq 1$ of the real axis. The von Neumann stability condition is satisfied if $1 - 8r \geq -1$, i.e., when $r \leq 1/4$.

Example 9

In addition to the previous Example 5, let us now consider another example of the scheme that connects the values of the difference solution on the three, rather than two, consecutive time levels of the grid.

A Cauchy problem for the one-dimensional homogeneous d'Alembert (wave) equation:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0, \quad -\infty < x < \infty, \quad 0 < t \leq T,$$
$$u(x,0) = \psi^{(0)}(x), \quad \frac{\partial u(x,0)}{\partial t} = \psi^{(1)}(x), \quad -\infty < x < \infty,$$

can be approximated on a uniform Cartesian grid by the following scheme:

$$\frac{u_m^{p+1} - 2u_m^p + u_m^{p-1}}{\tau^2} - \frac{u_{m+1}^p - 2u_m^p + u_{m-1}^p}{h^2} = 0,$$
$$u_m^0 = \psi_m^{(0)}, \quad \frac{u_m^1 - u_m^0}{h} = \psi_m^{(1)},$$
$$m = 0, \pm 1, \pm 2, \ldots, \quad p = 1, 2, \ldots, [T/\tau] - 1.$$

Substituting a solution of type (10.76) into the foregoing finite-difference equation, we obtain the following quadratic equation for determining $\lambda = \lambda(\alpha)$:

$$\lambda^2 - 2\left(1 - 2r^2 \sin^2 \frac{\alpha}{2}\right)\lambda + 1 = 0, \quad r = \frac{\tau}{h}.$$

The product of the two roots of this equation is equal to one. If its discriminant:

$$D(\alpha) = 4r^2 \sin^2 \frac{\alpha}{2}\left(r^2 \sin^2 \frac{\alpha}{2} - 1\right)$$

is negative, then the roots $\lambda_1(\alpha)$ and $\lambda_2(\alpha)$ are complex conjugate and both have a unit modulus. If $r < 1$, the discriminant $D(\alpha)$ remains negative for all $\alpha \in [0, 2\pi)$.

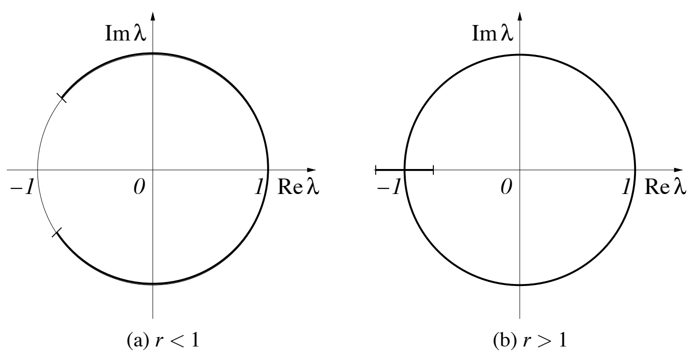

FIGURE 10.9: Spectrum of the scheme for the wave equation.

The spectrum for this case is shown in Figure 10.9(a); it fills an arc of the unit circle. If $r = 1$, the spectrum fills exactly the entire unit circle. When $r > 1$, the discriminant $D(\alpha)$ may be either negative or positive depending on the value of α. In this case, once the argument α increases from 0 to π the roots $\lambda_1(\alpha)$ and $\lambda_2(\alpha)$ depart from the point $\lambda = 1$ and move along the unit circle: One root moves clockwise and the other root counterclockwise, and then they merge at $\lambda = -1$. After that one root moves away from this point along the real axis to the right, and the other one to the left, because they are both real and their product $\lambda_1\lambda_2 = 1$, see Figure 10.9(b). The von Neumann stability condition is met for $r \leq 1$.

Consider a Cauchy problem for the following first order hyperbolic system of equations that describes the propagation of acoustic waves:

$$\begin{gathered}\frac{\partial v}{\partial t} = \frac{\partial w}{\partial x}, \quad \frac{\partial w}{\partial t} = \frac{\partial v}{\partial x}, \\ -\infty < x < \infty, \quad 0 < t \le T, \\ v(x,0) = \psi^{(1)}(x), \quad w(x,0) = \psi^{(2)}(x).\end{gathered} \tag{10.89a}$$

Let us set:

$$\boldsymbol{u}(x,t) = \begin{bmatrix} v(x,t) \\ w(x,t) \end{bmatrix}, \quad \boldsymbol{\psi}(x) = \begin{bmatrix} \psi^{(1)}(x) \\ \psi^{(2)}(x) \end{bmatrix}.$$

Then, problem (10.89a) can be recast in the matrix form:

$$\begin{gathered}\frac{\partial \boldsymbol{u}}{\partial t} - \boldsymbol{A}\frac{\partial \boldsymbol{u}}{\partial x} = \boldsymbol{0}, \quad -\infty < x < \infty, \quad 0 < t \le T, \\ \boldsymbol{u}(x,0) = \boldsymbol{\psi}(x), \quad -\infty < x < \infty,\end{gathered} \tag{10.89b}$$

where

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

We will now analyze two difference schemes that approximate problem (10.89b).

Example 10

Consider the scheme:

$$\frac{\boldsymbol{u}_m^{p+1} - \boldsymbol{u}_m^p}{\tau} - \boldsymbol{A}\frac{\boldsymbol{u}_{m+1}^p - \boldsymbol{u}_m^p}{h} = 0, \quad \boldsymbol{u}_m^0 = \boldsymbol{\psi}_m. \tag{10.90}$$

We will be looking for a solution to the vector finite-difference equation (10.90) in the form (10.80):

$$\boldsymbol{u}_m^p = \lambda^p \begin{bmatrix} v^0 \\ w^0 \end{bmatrix} e^{i\alpha m}.$$

Substituting this expression into equation (10.90) we obtain:

$$\frac{\lambda - 1}{\tau}\boldsymbol{u}^0 - \boldsymbol{A}\frac{e^{i\alpha} - 1}{h}\boldsymbol{u}^0 = \boldsymbol{0},$$

or alternatively,

$$(\lambda - 1)\boldsymbol{u}^0 - r(e^{i\alpha} - 1)\boldsymbol{A}\boldsymbol{u}^0 = \boldsymbol{0}, \quad r = \tau/h. \tag{10.91}$$

Equality (10.91) can be considered as a vector form of the system of linear algebraic equations with respect to the components of the vector $\boldsymbol{u}^0$. System (10.91) can be written as:

$$\begin{bmatrix} \lambda - 1 & -r(e^{i\alpha} - 1) \\ -r(e^{i\alpha} - 1) & \lambda - 1 \end{bmatrix} \begin{bmatrix} v^0 \\ w^0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{10.92}$$

System (10.92) may only have a nontrivial solution $\boldsymbol{u}^0 = \begin{bmatrix} v^0 \\ w^0 \end{bmatrix}$ if its determinant turns into zero, which yields the following equation for $\lambda = \lambda(\alpha)$:

$$(\lambda - 1)^2 = r^2(e^{i\alpha} - 1)^2.$$

Consequently,

$$\lambda_1(\alpha) = 1 - r + re^{i\alpha},$$
$$\lambda_2(\alpha) = 1 + r - re^{i\alpha}.$$

When the parameter α varies between 0 and 2π, the roots $\lambda_1(\alpha)$ and $\lambda_1(\alpha)$ sweep two circles of radius r centered at the locations $1-r$ and $1+r$, respectively. Therefore, the von Neumann stability conditions may never be satisfied; irrespective of any particular value of r.

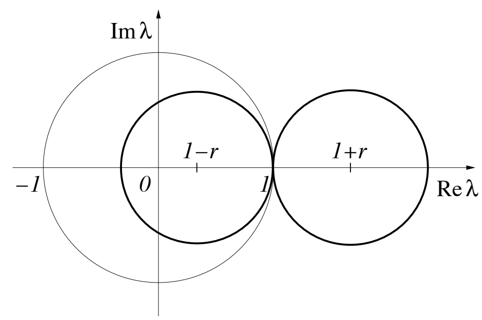

FIGURE 10.10: Spectrum of scheme (10.90).

Example 11

Consider the vector Lax-Wendroff scheme:

$$\begin{aligned} &\frac{\boldsymbol{u}_m^{p+1} - \boldsymbol{u}_m^p}{\tau} - \boldsymbol{A}\frac{u_{m+1}^p - u_{m-1}^p}{2h} - \frac{\tau}{2h^2}\boldsymbol{A}^2(\boldsymbol{u}_{m+1}^p - 2\boldsymbol{u}_m^p + \boldsymbol{u}_{m-1}^p) = \boldsymbol{0}, \\ &\boldsymbol{u}_m^0 = \boldsymbol{\psi}_m, \quad m = 0, \pm 1, \pm 2, \ldots, \quad p = 0, 1, \ldots, [T/\tau] - 1, \end{aligned} \tag{10.93}$$

that approximates problem (10.89b) on its smooth solutions with second order accuracy and that is analogous to the scalar scheme (10.83) for the Cauchy problem (10.82). As in Example 10, the finite-difference equation of (10.93) may only have a non-trivial solution of type (10.80) if the determinant of the corresponding linear system for finding $\boldsymbol{u}^0 = \begin{bmatrix} v^0 \\ w^0 \end{bmatrix}$ turns into zero.

Writing down this determinant and requiring that it be equal to zero, we obtain a quadratic equation with respect to $\lambda = \lambda(\alpha)$. Its roots can be easily found:

$$\lambda_1(\alpha) = 1 + ir\sin\alpha - 2r^2\sin^2\frac{\alpha}{2},$$
$$\lambda_2(\alpha) = 1 - ir\sin\alpha - 2r^2\sin^2\frac{\alpha}{2}.$$

These formulae are analogous to those obtained for the scalar Example 2, and similarly to that example we have:

$$1 - |\lambda_{1,2}(\alpha)|^2 = 4r^2\sin^4\frac{\alpha}{2}(1 - r^2).$$

We therefore see that the spectrum $\{\lambda_1(\alpha), \lambda_2(\alpha)\}$ belongs to the unit disk if $r \leq 1$.

10.3.4 Stability in C

Let us emphasize that the type of stability we have analyzed in Sections 10.3.1–10.3.3 was stability in the sense of the maximum norm (10.71). Alternatively, it is referred to as stability in (the space) C. This space contains all bounded numerical sequences. The von Neumann spectral condition (10.78) is *necessary* for the scheme to be stable in C. As far as the sufficient conditions, in some simple cases stability in C can be proved directly, for example, using maximum principle, as done in Section 10.1.3 for the first order explicit upwind scheme and in Section 10.6.1 for an explicit scheme for the heat equation. Otherwise, sufficient conditions for stability in C turn out to be delicate and may require rather sophisticated arguments. The analysis of a general case even for one scalar constant coefficient difference equation goes beyond the scope of the current book, and we refer the reader to the original work by Fedoryuk [Fed67] (see also his monograph [Fed77, Chapter V, § 4]). In addition, in [RM67, Chapter 5] the reader can find an account of the work by Strang and by Thomee on the subject.

10.3.5 Sufficiency of the Spectral Stability Condition in l_2

However, *a sufficient condition for stability* may sometimes be easier to find if we were to use a different norm instead of the maximum norm (10.71). Let, for example,

$$\begin{gathered}
\|u^p\|^2 = \sum_{m=-\infty}^{\infty} |u_m^p|^2, \quad \|\varphi^p\|^2 = \sum_{m=-\infty}^{\infty} |\varphi_m^p|^2, \quad \|\psi\|^2 = \sum_{m=-\infty}^{\infty} |\psi_m|^2, \\
\|u^{(h)}\|_{U_h} = \max_p \|u^p\|, \quad \|f^{(h)}\|_{F_h} = \left\| \begin{matrix} \varphi^p \\ \psi \end{matrix} \right\|_{F_h} = \max\{\|\psi\|, \max_p \|\varphi^p\|\}.
\end{gathered} \tag{10.94}$$

Relations (10.94) define Euclidean (i.e., l_2) norms for u^p, φ^p, and ψ. Accordingly, stability in the sense of the norms given by (10.94) is referred to as stability in (the space) l_2. We recall that the space l_2 is a Hilbert space of all numerical sequences, for which the sum of squares of absolute values of all their terms is bounded.

Consider a general constant coefficient finite-difference Cauchy problem:

$$\begin{gathered}
\sum_{j=-j_{\text{left}}}^{j_{\text{right}}} b_j u_{m+j}^{p+1} - \sum_{j=-j_{\text{left}}}^{j_{\text{right}}} a_j u_{m+j}^p = \varphi_m^p, \\
u_m^0 = \psi_m, \quad m = 0, \pm 1, \pm 2, \dots, \quad p = 0, 1, \dots, [T/\tau] - 1,
\end{gathered} \tag{10.95}$$

under the assumption that

$$\sum_{j=-j_{\text{left}}}^{j_{\text{right}}} b_j e^{i\alpha j} \neq 0, \quad 0 \leq \alpha < 2\pi.$$

Note that all spatially one-dimensional schemes from Section 10.3.3 fit into the category (10.95), even for a relatively narrow range: $j_{\text{left}} = j_{\text{right}} = 1$.

THEOREM 10.3
For the scheme (10.95) to be stable in l_2 with respect to the initial data, i.e., for the following inequality to hold:

$$\|u^p\| \le c\|\psi\|, \quad p = 0, 1, \ldots, [T/\tau], \tag{10.96}$$

where the constant c does not depend on h [or on $\tau = \tau(h)$], is is necessary and sufficient that the von Neumann condition (10.78) be satisfied, i.e., that the spectrum of the scheme $\lambda = \lambda(\alpha)$ belong to the disk:

$$|\lambda(\alpha)| \le 1 + c_1\tau, \tag{10.97}$$

where c_1 is another constant that does not depend either on α or on τ.

PROOF We will first prove the sufficiency. By hypotheses of the theorem, the number series $\sum_{m=-\infty}^{\infty} |\psi_m|^2$ converges. Then, the function series of the independent variable α:

$$\sum_{m=-\infty}^{\infty} \psi_m e^{-i\alpha m}$$

also converges in the space $L_2[0, 2\pi]$, and its sum that we denote $\Psi(\alpha)$, $0 \le \alpha \le 2\pi$, is a function that has ψ_m as the Fourier coefficients:

$$\psi_m = \frac{1}{2\pi}\int_0^{2\pi} \Psi(\alpha) e^{i\alpha m} d\alpha, \quad m = 0, \pm 1, \pm 2, \ldots, \tag{10.98}$$

(a realization of the Riesz-Fischer theorem, see, e.g., [KF75, Section 16]). Moreover, the following relation holds:

$$\|\psi\|^2 = \sum_{m=-\infty}^{\infty} |\psi_m|^2 = \frac{1}{2\pi}\int_0^{2\pi} |\Psi(\alpha)|^2 d\alpha = \frac{1}{2\pi}\|\Psi\|_2^2$$

known as the Parseval equality.

Consider a homogeneous counterpart to the difference equation (10.95):

$$\sum_{j=-j_{\text{left}}}^{j_{\text{right}}} b_j u_{m+j}^{p+1} - \sum_{j=-j_{\text{left}}}^{j_{\text{right}}} a_j u_{m+j}^{p} = 0,$$
$$m = 0, \pm 1, \pm 2, \ldots, \quad p = 0, 1, \ldots, [T/\tau] - 1,$$

For any $\alpha \in [0, 2\pi)$ this equation obviously has a solution of the type:

$$u_m^p(\alpha) = \lambda^p(\alpha) e^{i\alpha m} \tag{10.99}$$

for some particular $\lambda = \lambda(\alpha)$ that can be determined by substitution:

$$\lambda(\alpha) = \left(\sum_{j=-j_{\text{left}}}^{j_{\text{right}}} a_j e^{i\alpha j} \right) \left(\sum_{j=-j_{\text{left}}}^{j_{\text{right}}} b_j e^{i\alpha j} \right)^{-1}.$$

Then, the grid function:

$$u_m^p = \frac{1}{2\pi} \int_0^{2\pi} \Psi(\alpha) \lambda^p(\alpha) e^{i\alpha m} d\alpha, \quad m = 0, \pm 1, \pm 2, \ldots, \tag{10.100}$$

provides solution to the Cauchy problem (10.95) for the case $\varphi_m^p = 0$ because it is a linear combination of solutions $u_m^p(\alpha)$ of (10.99), and coincides with ψ_m for $p = 0$, see (10.98). Note that the integral on the right-hand side of formula (10.100) converges by virtue of the Parseval equality, because $\int_0^{2\pi} |\Psi(\alpha)|^2 d\alpha < \infty \Longrightarrow \int_0^{2\pi} |\Psi(\alpha)| d\alpha < \infty$.

If the von Neumann spectral condition (10.97) is satisfied, then

$$|\lambda(\alpha)|^p \leq |1 + c_1 \tau|^{T/\tau} \leq e^{c_1 T}. \tag{10.101}$$

Consequently, using representation (10.100) along with the Parseval equality and inequality (10.101), we can obtain:

$$\|u^p\|^2 = \frac{1}{2\pi} \int_0^{2\pi} |\lambda^p(\alpha) \Psi(\alpha)|^2 d\alpha \leq e^{2c_1 T} \frac{1}{2\pi} \int_0^{2\pi} |\Psi(\alpha)|^2 d\alpha = e^{2c_1 T} \|\psi\|^2,$$

which clearly implies stability with respect to the initial data: $\|u^p\| \leq c\|\psi\|$.

To prove the necessity, we will need to show that if (10.97) holds for no fixed c_1, then the scheme is unstable. We should emphasize that to demonstrate the instability for the chosen norm (10.94) we may not exploit the unboundedness of the solution $u_m^p = \lambda^p(\alpha) e^{i\alpha m}$ that takes place in this case, because the grid function $\{e^{i\alpha m}\}$ does not belong to l_2.

Rather, let us take a particular $\Psi(\alpha) \in L_2[0, 2\pi]$ such that

$$\frac{1}{2\pi} \int_0^{2\pi} |\lambda(\alpha)|^{2p} |\Psi(\alpha)|^2 d\alpha \geq \max_\alpha \left(|\lambda(\alpha)|^{2p} - \varepsilon \right) \frac{1}{2\pi} \int_0^{2\pi} |\Psi(\alpha)|^2 d\alpha, \tag{10.102}$$

where $\varepsilon > 0$ is given. For an arbitrary ε, estimate (10.102) can always be guaranteed by selecting:

$$\Psi(\alpha) = \begin{cases} 1, & \text{if} \quad \alpha \in [\alpha^* - \delta, \alpha^* + \delta], \\ 0, & \text{if} \quad \alpha \notin [\alpha^* - \delta, \alpha^* + \delta], \end{cases}$$

where $\alpha^* = \arg\max_\alpha |\lambda(\alpha)|$ and $\delta > 0$. Indeed, as the function $|\lambda(\alpha)|^{2p}$ is continuous, inequality (10.102) will hold for a sufficiently small $\delta = \delta(\varepsilon)$.

If estimate (10.101) does not take place, then we can find a sequence h_k, $k=0,1,2,3,\ldots$, and the corresponding sequence $\tau_k=\tau(h_k)$ such that

$$c_k = \left(\max_{\alpha} |\lambda(\alpha,h_k)|\right)^{[T/\tau_k]} \longrightarrow \infty \quad \text{as} \quad k \longrightarrow \infty.$$

Let us set $\varepsilon=1$ and choose $\Psi(\alpha)$ to satisfy (10.102). Define ψ_m as Fourier coefficients of the function $\Psi(\alpha)$, according to formula (10.98). Then, inequality (10.102) for $p_k=[T/\tau_k]$ transforms into:

$$\|u^{p_k}\|^2 \geq (c_k^2-1)\|\psi\|^2 \Longrightarrow \|u^{p_k}\| \geq (c_k-1)\|\psi\|,$$
$$c_k \longrightarrow \infty \quad \text{as} \quad k \longrightarrow \infty,$$

i.e., there is indeed no stability (10.96) with respect to the initial data. ∎

Theorem 10.3 establishes *equivalence* between the von Neumann spectral condition and the l_2 stability of scheme (10.95) with respect to the initial data. In fact, one can go even further and prove that the von Neumann spectral condition is *necessary and sufficient* for the full-fledged l_2 stability of the scheme (10.95) as well, i.e., when the right-hand side φ_m^p is not disregarded. One implication, the necessity, immediately follows from Theorem 10.3, because if the von Neumann condition does not hold, then the scheme is unstable even with respect to the initial data. The proof of the other implication, the sufficiency, can be found in [GR87, § 25]. This proof is based on using the discrete Green's functions. In general, once stability with respect to the initial data has been established, stability of the full inhomogeneous problem can be derived using the Duhamel principle. This principle basically says that the solution to the inhomogeneous problem can be obtained as linear superposition of the solutions to some specially chosen homogeneous problems. Consequently, a stability estimate for the inhomogeneous problem can be obtained on the basis of stability estimates for a series of homogeneous problems, see [Str04, Chapter 9].

10.3.6 Scalar Equations vs. Systems

As of yet, our analysis of finite-difference stability has focused mostly on scalar equations; we have considered a 2×2 system only in Examples 10 and 11 of Section 10.3.3. In Examples 5 and 9, we have also considered scalar difference equations that connect the values of the solution on more than two consecutive time levels; those can be reduced to systems on two time levels.

In general, a constant coefficient finite-difference Cauchy problem with vector unknowns (i.e., a system) can be written in the form similar to (10.95):

$$\sum_{j=-j_{\text{left}}}^{j_{\text{right}}} \boldsymbol{B}_j \boldsymbol{u}_{m+j}^{p+1} - \sum_{j=-j_{\text{left}}}^{j_{\text{right}}} \boldsymbol{A}_j \boldsymbol{u}_{m+j}^{p} = \boldsymbol{\varphi}_m^p, \tag{10.103}$$
$$\boldsymbol{u}_m^0 = \boldsymbol{\psi}_m, \quad m=0,\pm1,\pm2,\ldots, \quad p=0,1,\ldots,[T/\tau]-1,$$

under the assumption that the matrices

$$\sum_{j=-j_{\text{left}}}^{j_{\text{right}}} \boldsymbol{B}_j e^{i\alpha j}, \quad 0 \le \alpha < 2\pi,$$

are non-singular. In formula (10.103), $\boldsymbol{u}_m^p$, $\boldsymbol{\varphi}_m^p$, and $\boldsymbol{\psi}_m$ are grid vector functions of a fixed dimension, and $\boldsymbol{A}_j = \boldsymbol{A}_j(h), \boldsymbol{B}_j = \boldsymbol{B}_j(h)$, $j = -j_{\text{left}}, \dots, j_{\text{right}}$, are given square matrices of the same dimension.

Solution to the homogeneous counterpart of equation (10.103) can be sought for in the form (10.80), where $\boldsymbol{u}^0 = \boldsymbol{u}^0(\alpha, h)$ and $\lambda = \lambda(\alpha, h)$ are the eigenvectors and eigenvalues, respectively, of the amplification matrix of scheme (10.103):

$$\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\alpha, h) = \left(\sum_{j=-j_{\text{left}}}^{j_{\text{right}}} \boldsymbol{B}_j e^{i\alpha j} \right)^{-1} \left(\sum_{j=-j_{\text{left}}}^{j_{\text{right}}} \boldsymbol{A}_j e^{i\alpha j} \right) \tag{10.104}$$

The von Neumann spectral condition (10.97) is clearly necessary for stability of finite-difference systems in all norms. Indeed, if it is not met, then estimate (10.101) will not hold, and the scheme will develop a catastrophic exponential instability that cannot be fixed by any reasonable choice of norms.

Yet the von Neumann condition remains *only a necessary* stability condition for systems in either C or l_2. For C, the analysis of sufficient conditions becomes cumbersome already for the scalar case (see Section 10.3.4), and even in l_2 obtaining sufficient conditions for systems proves rather involved.

Qualitatively, the difficulties stem from the fact that the amplification matrix (10.104) may have multiple eigenvalues and as a consequence, may not necessarily have a full set of eigenvectors. If a multiple eigenvalue occurs exactly on the unit circle or just outside the unit disk, this may still cause instability even when all the eigenvalues satisfy the von Neumann constraint (10.97) (similar to Theorem 9.2).

An example is provided by the leap-frog scheme (10.86). If $r = 1$ and $\alpha = \pi/2$, then we have $\lambda_{1,2} = i$, and if $r = 1$ and $\alpha = 3\pi/2$, then $\lambda_{1,2} = -i$. In either case, in addition to (10.80) there will be a solution of the form $\boldsymbol{u}_m^p = p\lambda^p \left[\boldsymbol{u}^0 e^{i\alpha m}\right]$, which is a manifestation of a gradually (linearly) developing instability.

Of course, if the amplification matrix appears normal (a matrix that commutes with its adjoint) and therefore unitarily diagonalizable, then none of the aforementioned difficulties is present, and the von Neumann condition becomes not only necessary but also sufficient for stability of the vector scheme (10.103) in l_2.

Otherwise, the question of stability for scheme (10.103) can be *equivalently reformulated* using the new concept of stability for families of matrices. A family of square matrices (of a given fixed dimension) is said to be stable if there is a constant $K > 0$ such that for any particular matrix $\boldsymbol{\Lambda}$ from the family, and any positive integer p, the following estimate holds: $\|\boldsymbol{\Lambda}^p\| \le K$. Scheme (10.103) is stable in l_2 if and only if the family of amplification matrices $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\alpha, h)$ given by (10.104) is stable in the sense of the previous definition (this family is parameterized by $\alpha \in [0, 2\pi)$ and $h > 0$). Theorem 10.4, known as the Kreiss matrix theorem, provides some necessary and sufficient conditions for a family of matrices to be stable.

THEOREM 10.4 (Kreiss)
Stability of a family of matrices $\mathbf{\Lambda}$ is equivalent to any of the following:

1. *There is a constant $C_1 > 0$ such that for any matrix $\mathbf{\Lambda}$ from the given family, and any complex number z, $|z| > 1$, there is a resolvent $(\mathbf{\Lambda} - z\boldsymbol{I})^{-1}$ bounded as:*
$$\left\|(\mathbf{\Lambda} - z\boldsymbol{I})^{-1}\right\| \leq \frac{C_1}{|z|-1}.$$
2. *There are constants $C_2 > 0$ and $C_3 > 0$, and for any matrix $\mathbf{\Lambda}$ from the given family there is a non-singular matrix $\boldsymbol{M}$ such that $\|\boldsymbol{M}\| \leq C_2$, $\|\boldsymbol{M}^{-1}\| \leq C_2$, and the matrix $\boldsymbol{D} \stackrel{\text{def}}{=} \boldsymbol{M}\mathbf{\Lambda}\boldsymbol{M}^{-1}$ is upper triangular, with the off-diagonal entries that satisfy:*
$$|d_{ij}| \leq C_3 \min\{1-\kappa_i, 1-\kappa_j\},$$
where $\kappa_i = d_{ii}$ and $\kappa_j = d_{jj}$ are the corresponding diagonal entries of $\boldsymbol{D}$, i.e., the eigenvalues of $\mathbf{\Lambda}$.
3. *There is a constant $C_4 > 0$, and for any matrix $\mathbf{\Lambda}$ from the given family there is a Hermitian positive definite matrix $\boldsymbol{H}$, such that*
$$C_4^{-1}\boldsymbol{I} \leq \boldsymbol{H} \leq C_4\boldsymbol{I} \quad \text{and} \quad \mathbf{\Lambda}^*\boldsymbol{H}\mathbf{\Lambda} \leq \boldsymbol{H}.$$

The proof can be found in [RM67, Chapter 4] or [Str04, Chapter 9].

Exercises

1. Consider the so-called weighted scheme for the heat equation:
$$\frac{u_m^{p+1} - u_m^p}{\tau} = \sigma \frac{u_{m+1}^{p+1} - 2u_m^{p+1} + u_{m-1}^{p+1}}{h^2} + (1-\sigma)\frac{u_{m+1}^p - 2u_m^p + u_{m-1}^p}{h^2},$$
$$u_m^0 = \psi_m, \quad m = 0, \pm 1, \pm 2, \ldots, \quad p = 0, 1, \ldots, [T/\tau] - 1,$$
where the real parameter $\sigma \in [0,1]$ is called the weight (between the fully explicit scheme, $\sigma = 0$, and fully implicit scheme, $\sigma = 1$). What values of σ guarantee that the scheme will meet the von Neumann stability condition for any $r = \tau/h^2 = \text{const}$?
2. Consider the Cauchy problem (10.87) for the heat equation. The scheme:
$$\frac{u_m^{p+1} - u_m^{p-1}}{2\tau} - a^2 \frac{u_{m+1}^p - 2u_m^p + u_{m-1}^p}{h^2} = \varphi_m^p,$$
$$u_m^0 = \psi_m, \quad u_m^1 = \tilde{\psi}_m, \quad m = 0, \pm 1, \pm 2, \ldots, \quad p = 0, 1, \ldots, [T/\tau] - 1,$$
where we assume that $\varphi(x,0) \equiv 0$ and define:
$$\tilde{\psi}_m = u(x,0) + \tau \frac{\partial u(x,0)}{\partial t}\bigg|_{x=x_m} = u(x,0) + \tau a^2 \frac{\partial^2 u(x,0)}{\partial x^2}\bigg|_{x=x_m} = \psi_m + \tau a^2 \psi''(x_m),$$
approximates problem (10.87) on its smooth solutions with accuracy $\mathscr{O}(\tau^2 + h^2)$. Does this scheme satisfy the von Neumann spectral stability condition for $r = \tau/h^2 = \text{const}$?

3. For the two-dimensional Cauchy problem:

$$\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = \varphi(x,y,t), \quad -\infty < x,\, y < \infty, \quad 0 < t \le T,$$
$$u(x,y,0) = \psi(x,y), \quad -\infty < x,\, y < \infty,$$

investigate the von Neumann spectral stability of:

a) The first order explicit scheme:

$$\frac{u^{p+1}_{m,n} - u^p_{m,n}}{\tau} - \frac{u^p_{m+1,n} - u^p_{m,n}}{h} - \frac{u^p_{m,n+1} - u^p_{m,n}}{h} = \varphi^p_{m,n},$$
$$u^0_{m,n} = \psi_{m,n}, \quad m,\, n = 0, \pm 1, \pm 2, \dots, \quad p = 0, 1, \dots, [T/\tau] - 1;$$

b) The second order explicit scheme:

$$\frac{u^{p+1}_{m,n} - u^{p-1}_{m,n}}{2\tau} - \frac{u^p_{m+1,n} - u^p_{m-1,n}}{2h} - \frac{u^p_{m,n+1} - u^p_{m,n-1}}{2h} = \varphi^p_{m,n},$$
$$u^0_{m,n} = \psi_{m,n}, \quad u^1_{m,n} = \psi_{m,n} + \tau[\psi'_x(x_m,y_n) + \psi'_y(x_m,y_n) + \varphi(x_m,y_n,0)],$$
$$m,\, n = 0, \pm 1, \pm 2, \dots, \quad p = 0, 1, \dots, [T/\tau] - 1.$$

4. Investigate the von Neumann spectral stability of the implicit two-dimensional scheme for the homogeneous heat equation:

$$\frac{u^{p+1}_{m,n} - u^p_{m,n}}{\tau} = \frac{u^{p+1}_{m+1,n} - 2u^{p+1}_{m,n} + u^{p+1}_{m-1,n}}{h^2} + \frac{u^{p+1}_{m,n+1} - 2u^{p+1}_{m,n} + u^{p+1}_{m,n-1}}{h^2}$$
$$u^0_{m,n} = \psi_{m,n}, \quad m, n = 0, \pm 1, \pm 2, \dots, \quad p = 0, 1, \dots, [T/\tau] - 1.$$

5. Investigate the von Neumann stability of the implicit upwind scheme for the Cauchy problem (10.82):

$$\frac{u^{p+1}_m - u^p_m}{\tau} - \frac{u^{p+1}_{m+1} - u^{p+1}_m}{h} = \varphi^p_m, \tag{10.105}$$
$$u^0_m = \psi_m, \quad m = 0, \pm 1, \pm 2, \dots, \quad p = 0, 1, \dots, [T/\tau] - 1.$$

6. Investigate the von Neumann stability of the implicit downwind scheme for the Cauchy problem (10.82):

$$\frac{u^{p+1}_m - u^p_m}{\tau} - \frac{u^{p+1}_m - u^{p+1}_{m-1}}{h} = \varphi^p_m, \tag{10.106}$$
$$u^0_m = \psi_m, \quad m = 0, \pm 1, \pm 2, \dots, \quad p = 0, 1, \dots, [T/\tau] - 1.$$

7. Investigate the von Neumann stability of the implicit central scheme for the Cauchy problem (10.82):

$$\frac{u^{p+1}_m - u^p_m}{\tau} - \frac{u^{p+1}_{m+1} - u^{p+1}_{m-1}}{2h} = \varphi^p_m, \tag{10.107}$$
$$u^0_m = \psi_m, \quad m = 0, \pm 1, \pm 2, \dots, \quad p = 0, 1, \dots, [T/\tau] - 1.$$

8.* Transform the leap-frog scheme (10.86) of Example 5, Section 10.3.3, and the central-difference scheme for the d'Alembert equation of Example 9, Section 10.3.3, to the schemes written for finite-difference systems, as opposed to scalar equations, but connecting only two, as opposed to three, consecutive time levels of the grid. Investigate the von Neumann stability by calculating spectra of the corresponding amplification matrices (10.104).

Hint. Use the difference $\{u_m^{p+1} - u_m^p\}$ as the second unknown grid function.

10.4 Stability for Problems with Variable Coefficients

The von Neumann necessary condition that we have introduced in Section 10.3 to analyze stability of linear finite-difference Cauchy problems with constant coefficients can, in fact, be applied to a wider class of formulations. A simple extension that we describe in this section allows one to exploit the von Neumann condition to analyze stability of problems with variable coefficients (continuous, but not necessarily constant) and even some nonlinear problems.

10.4.1 The Principle of Frozen Coefficients

Introduce a uniform Cartesian grid: $x_m = mh$, $m = 0, \pm 1, \pm 2, \dots$, $t_p = p\tau$, $p = 0, 1, 2, \dots$, and consider a finite-difference Cauchy problem for the homogeneous heat equation with the variable coefficient of heat conduction $a = a(x,t)$:

$$\begin{gathered} \frac{u_m^{p+1} - u_m^p}{\tau} - a(x_m, t_p) \frac{u_{m+1}^p - 2u_m^p + u_{m-1}^p}{h^2} = 0, \\ u_m^0 = \psi(x_m), \quad m = 0, \pm 1, \pm 2, \dots, \quad p \geq 0. \end{gathered} \tag{10.108}$$

Next, take an arbitrary point $(\tilde{x}, \tilde{t})$ in the domain of problem (10.108) and "freeze" the coefficients of problem (10.108) at this point. Then, we arrive at the constant-coefficient finite-difference equation:

$$\begin{gathered} \frac{u_m^{p+1} - u_m^p}{\tau} - a(\tilde{x}, \tilde{t}) \frac{u_{m+1}^p - 2u_m^p + u_{m-1}^p}{h^2} = 0, \\ m = 0, \pm 1, \pm 2, \dots, \quad p \geq 0. \end{gathered} \tag{10.109}$$

Having obtained equation (10.109), we can formulate the following principle of frozen coefficients. *For the original variable-coefficient Cauchy problem (10.108) to be stable it is necessary that the constant-coefficient Cauchy problem for the difference equation (10.109) satisfy the von Neumann spectral stability condition.*

To justify the principle of frozen coefficients, we will provide an heuristic argument rather than a proof. When the grid is refined, the variation of the coefficient $a(x,t)$ in a neighborhood of the point $(\tilde{x}, \tilde{t})$ becomes smaller if measured over any