Let us then define the sequence $\mu_k \stackrel{\text{def}}{=} \left(\frac{L}{2\pi}\right)^{r+1}\sqrt{A_k^2+B_k^2}$. Inequality (3.44) implies that the series $\sum\limits_{k=1}^{\infty}\mu_k^2$ converges. Now assume for definiteness that equalities (3.42) hold; then we have:

$$|\alpha_k| = \left(\frac{L}{2\pi}\right)^{r+1}\frac{|A_k|}{k^{r+1}} \le \left(\frac{L}{2\pi}\right)^{r+1}\frac{\sqrt{A_k^2+B_k^2}}{k^{r+1}} = \frac{\mu_k}{k^{r+1}},$$

and similarly

$$|\beta_k| = \left(\frac{L}{2\pi}\right)^{r+1}\frac{|B_k|}{k^{r+1}} \le \left(\frac{L}{2\pi}\right)^{r+1}\frac{\sqrt{A_k^2+B_k^2}}{k^{r+1}} = \frac{\mu_k}{k^{r+1}}.$$

The same estimates can obviously be obtained when relations (3.43) hold instead of (3.42). We can therefore say that the Fourier coefficients (3.40) satisfy the following inequalities:

$$|\alpha_k| \le \frac{\mu_k}{k^{r+1}}, \qquad |\beta_k| \le \frac{\mu_k}{k^{r+1}}, \tag{3.45}$$

where the sequence $\mu_k = o(1)$ is such that the series $\sum\limits_{k=1}^{\infty}\mu_k^2$ converges.

Then, for the remainder (3.39) of the Fourier series we can write:

$$\begin{aligned}
|\delta S_n(x)| &= \left|\sum_{k=n+1}^{\infty}\alpha_k\cos\frac{2\pi kx}{L} + \sum_{k=n+2}^{\infty}\beta_k\sin\frac{2\pi kx}{L}\right| \\
&\le \sum_{k=n+1}^{\infty}|\alpha_k| + \sum_{k=n+2}^{\infty}|\beta_k| \le \sum_{k=n+1}^{\infty}|\alpha_k|+|\beta_k| \\
&\le 2\sum_{k=n+1}^{\infty}\frac{\mu_k}{k^{r+1}} \le 2\sqrt{\sum_{k=n+1}^{\infty}\mu_k^2}\sqrt{\sum_{k=n+1}^{\infty}\frac{1}{k^{2(r+1)}}},
\end{aligned} \tag{3.46}$$

where the last estimate in (3.46) is obtained using the classical Cauchy-Schwarz inequality (see [KF75, Section 5]):

$$\sum_{k=1}^{\infty}u_kv_k \le \sqrt{\sum_{k=1}^{\infty}u_k^2}\sqrt{\sum_{k=1}^{\infty}v_k^2},$$

which holds as long as $\sum\limits_{k=1}^{\infty}u_k^2 < \infty$ and $\sum\limits_{k=1}^{\infty}v_k^2 < \infty$.

Next, we note that for $\xi \in [k-1,k]$ the following inequality always holds: $\dfrac{1}{k^{2(r+1)}} \le \dfrac{1}{\xi^{2(r+1)}}$, and consequently, $\dfrac{1}{k^{2(r+1)}} \le \displaystyle\int_{k-1}^{k}\frac{d\xi}{\xi^{2(r+1)}}$. Therefore,

$$\sum_{k=n+1}^{\infty}\frac{1}{k^{2(r+1)}} \le \sum_{k=n+1}^{\infty}\int_{k-1}^{k}\frac{d\xi}{\xi^{2(r+1)}} = \int_{n}^{\infty}\frac{d\xi}{\xi^{2(r+1)}} = \frac{1}{(2r+1)n^{2r+1}}.$$

Substituting the latter estimate into inequality (3.46), we obtain:

$$|\delta S_n(x)| \le \frac{2}{\sqrt{2r+1}} \sqrt{\sum_{k=n+1}^{\infty} \mu_k^2} \frac{1}{\sqrt{n^{2r+1}}}. \tag{3.47}$$

It only remains to notice that $\tilde{\zeta}_n \stackrel{\text{def}}{=} \frac{2}{\sqrt{2r+1}} \sqrt{\sum_{k=n+1}^{\infty} \mu_k^2} = o(1)$, because the series $\sum_{k=1}^{\infty} \mu_k^2$ converges. Then, estimate (3.47) does imply (3.41).

Having justified estimate (3.41), we next notice that the partial sum $S_n(x)$ is, in fact, a trigonometric polynomial of type (3.6). Due to the uniqueness of the trigonometric interpolating polynomial, see Theorem 3.1, we then have

$$Q_n\left(\cos\frac{2\pi}{L}x, \sin\frac{2\pi}{L}x, S_n\right) = S_n(x). \tag{3.48}$$

Moreover, estimates (3.32), (3.33), and (3.41) together yield:

$$\left|Q_n\left(\cos\frac{2\pi}{L}x, \sin\frac{2\pi}{L}x, \delta S_n\right)\right| \le L_n|\delta S_n| \le L_n \frac{\tilde{\zeta}_n}{n^{r+\frac{1}{2}}} \le 4\frac{\zeta_n}{n^{r-\frac{1}{2}}}, \tag{3.49}$$

where $\zeta_n = o(1)$ as $n \longrightarrow \infty$. Finally, by combining (3.41), (3.48), and (3.49) we obtain the following estimate for the interpolation error $R_n(x)$:

$$\begin{aligned}
|R_n(x)| &= \left|f(x) - Q_n\left(\cos\frac{2\pi}{L}x, \sin\frac{2\pi}{L}x, f\right)\right| \\
&= \left|f(x) - Q_n\left(\cos\frac{2\pi}{L}x, \sin\frac{2\pi}{L}x, S_n\right) - Q_n\left(\cos\frac{2\pi}{L}x, \sin\frac{2\pi}{L}x, \delta S_n\right)\right| \\
&= \left|f(x) - S_n(x) - Q_n\left(\cos\frac{2\pi}{L}x, \sin\frac{2\pi}{L}x, \delta S_n\right)\right| \\
&= \left|\delta S_n(x) - Q_n\left(\cos\frac{2\pi}{L}x, \sin\frac{2\pi}{L}x, \delta S_n\right)\right| \\
&\le |\delta S_n(x)| + \left|Q_n\left(\cos\frac{2\pi}{L}x, \sin\frac{2\pi}{L}x, \delta S_n\right)\right| \\
&\le |\delta S_n(x)| + L_n|\delta S_n(x)| \le \frac{\tilde{\zeta}_n}{n^{r+\frac{1}{2}}} + 4\frac{\zeta_n}{n^{r-\frac{1}{2}}} \le \text{const} \cdot \frac{\zeta_n}{n^{r-\frac{1}{2}}},
\end{aligned}$$

which is obviously equivalent to the required estimate (3.37). ∎

We emphasize that the rate of convergence of the trigonometric interpolating polynomials established by estimate (3.37) automatically becomes faster for smoother interpolated functions $f(x)$. In this sense, trigonometric interpolation of periodic functions appears to be *not susceptible to the saturation by smoothness.* This is a remarkable difference compared to the case of algebraic interpolation (Chapter 2), when the convergence rate is limited by the degree of the polynomial.