the elements of the subspace $\overset{\circ}{W}{}^{(N)}$. In doing so, the accuracy of approximation is to be measured in the energy norm. A similar result also holds for the Galerkin method.

Of course, the question of how large the right-hand side of inequality (12.41) may actually be does not have a direct answer, because the solution $u = u(x,y)$ is not known ahead of time. Therefore, the best we can do for evaluating this right-hand side is first to assume that the solution u belongs to some class of functions $U \subset \overset{\circ}{W}$, and then try and narrow down this class as much as possible using all a priori information about the solution that is available. Often, the class U can be characterized in terms of smoothness, because a given degree of regularity of the data φ and ψ normally translates into a certain level of smoothness of the solution u. For example, recall that the solution of problem (12.15) was assumed twice continuously differentiable. In this case, we can say that U is the class of all functions that are equal to zero at Γ and also have continuous second derivatives on $\bar{\Omega}$.

Once we have identified the maximally narrow class of functions U that contains the solution u, we can write instead of estimate (12.41):

$$\|w_N - u\|_{\overset{\circ}{W}}^2 \leq \text{const} \cdot \sup_{v \in U} \inf_{w \in \overset{\circ}{W}{}^{(N)}} \left[\|w - v\|' \right]^2. \tag{12.42}$$

Regarding this inequality, a natural expectation is that the narrower the class U, the closer the value on the right-hand side of (12.42) to that on the right-side of (12.41).

Next, we realize that the value on the right-hand side of inequality (12.42) depends on the choice of the approximating space $\overset{\circ}{W}{}^{(N)}$, and the best possible value therefore corresponds to:

$$\kappa_N(U, \overset{\circ}{W}) \overset{\text{def}}{=} \inf_{\overset{\circ}{W}{}^{(N)} \subset \overset{\circ}{W}} \sup_{v \in U} \inf_{w \in \overset{\circ}{W}{}^{(N)}} \|w - v\|'. \tag{12.43}$$

The quantity $\kappa_N(U, \overset{\circ}{W})$ is called the N-dimensional Kolmogorov diameter of the set U with respect to the space $\overset{\circ}{W}$ in the sense of the energy norm $\|\cdot\|'$. We have first encountered this concept in Section 2.2.4, see formula (2.39) on page 41. Note that the norm in the definition of the Kolmogorov diameter is not squared, unlike in formula (12.42). Note also that the outermost minimization in (12.43) is performed with respect to the N-dimensional space $\overset{\circ}{W}{}^{(N)}$ as a whole, and its result does not depend on the choice of a specific basis (12.27) in $\overset{\circ}{W}{}^{(N)}$. One and the same space can be equipped with different bases.

Kolmogorov diameters and related concepts are widely used in the theory of approximation, as well as in many other branches of mathematics. In the context of finite elements they provide optimal, i.e., unimprovable, estimates for the convergence rate of the method. Once the space $\overset{\circ}{W}$, the norm $\|\cdot\|'$, and the subspace $U \subset \overset{\circ}{W}$ have been fixed, the diameter (12.43) depends on the dimension N. Then, the optimal convergence rate of a finite element approximation is determined by how rapidly the diameter $\kappa_N(U, \overset{\circ}{W})$ decays as $N \longrightarrow \infty$. Of course, in every given case

there is no guarantee that a particular choice of $\overset{\circ}{W}{}^{(N)}$ will yield the rate that would come anywhere close to the theoretical optimum provided by the Kolmogorov diameter $\kappa_N(U,\overset{\circ}{W})$. In other words, the rate of convergence for a specific Ritz or Galerkin approximation may be slower than optimalas $N \longrightarrow \infty$. Consequently, it is the skill and experience of a numerical analyst that are required for choosing the approximating space $\overset{\circ}{W}{}^{(N)}$ is such a way that the actual quantity:

$$\sup_{v\in U} \inf_{w\in \overset{\circ}{W}{}^{(N)}} \|w-v\|'$$

that controls the right-hand side of (12.42) would not be much larger than the Kolmogorov diameter $\kappa_N(U,\overset{\circ}{W})$ of (12.43).

In many particular situations, the Kolmogorov diameters have been computed. For example, for the setup analyzed in this section, when the space $\overset{\circ}{W}$ contains all piecewise continuously differentiable functions on $\bar{\Omega} \subset \mathbb{R}^2$ equal to zero at $\Gamma = \partial\Omega$, the class U contains all twice continuously differentiable functions from $\overset{\circ}{W}$, and the norm is the energy norm, we have:

$$\kappa_N(U,\overset{\circ}{W}) = \mathscr{O}\Big(\frac{1}{\sqrt{N}}\Big) \quad \text{as} \quad N \longrightarrow \infty. \tag{12.44}$$

Under certain additional conditions (non-restrictive), it is also possible to show that the piecewise linear elements constructed in Section 12.2.4 converge with the same asymptotic rate $\mathscr{O}(N^{-1/2})$ when N increases, see, e.g., [GR87, § 39]. As such, the piecewise linear finite elements guarantee the optimal convergence rate of the Ritz method for problem (12.15), assuming that nothing else can be said about the solution $u = u(x,y)$ except that it has continuous second derivatives on $\bar{\Omega}$.

Note that as we have a total of N grid nodes on a two-dimensional domain Ω, for a regular uniform grid it would have implied that the grid size is $h = \mathscr{O}(N^{-1/2})$. In other words, the convergence rate of the piecewise linear finite elements appears to be $\mathscr{O}(h)$, and the optimal unimprovable rate (12.44) is also $\mathscr{O}(h)$. At a first glance, this seems to be a deterioration of convergence compared, e.g., to the standard central-difference scheme of Section 12.1, which converges with the rate $\mathscr{O}(h^2)$. This, however, is not the case. In Section 12.1, we measured the convergence in the norm that did not contain the derivatives of the solution, whereas in this section we are using the Sobolev norm (12.14) that contains the first derivatives.

Finally, recall that the convergence rates can be improved by selecting the approximating space that would be right for the problem, as well as by narrowing down the class of functions U that contains the solution u. These two strategies can be combined and implemented together in the framework of one adaptive procedure. Adaptive methods represent a class of rapidly developing, modern and efficient, approaches to finite element approximations. Both the size/shape of the elements, as well as their order (beyond linear), can be controlled. Following a special (multigrid-type) algorithm, the elements are adapted dynamically, i.e., in the course of computation. For example, by refining the grid and/or increasing the order locally, these

elements can very accurately approximate sharp variations in the solution. In other words, as soon as those particular areas of sharp variation are identified inside Ω, the class U becomes narrower, and at the same time, the elements provide a better, "fine-tuned," basis for approximation. Convergence of these adaptive finite element approximations may achieve spectral rates. We refer the reader to the recent monograph by Demkowicz for further detail [Dem06].

To conclude this chapter, we will compare the method of finite elements with the method of finite differences from the standpoint of how convergence of each method is established. Recall, the study of convergence for finite-difference approximations consists of analyzing two independent properties, consistency and stability, see Theorem 10.1 on page 314. In the context of finite elements, consistency as defined in Section 10.1.2 or 12.1.1 (small truncation error on smooth solutions) is no longer needed for proving convergence. Instead, we need approximation of the class $U \ni u$ by the functions from $\overset{\circ}{W}{}^{(N)}$, see estimate (12.42). Stability for finite elements shall be understood as good conditioning (uniform with respect to N) of the Ritz system matrix (12.31) or of a similar Galerkin matrix, see (12.36). The ideal case here, which cannot normally be realized in practice, is when the basis $w_n^{(N)}$, $n = 1,2,\ldots,N$, is orthonormal. Then, the matrix $\boldsymbol{A}^{(N)}$ becomes a unit matrix. Stability still remains important when computing with finite elements, although not for justifying convergence, but for being able to disregard the small round-off errors.

Exercises

1. Prove that the quadratic form of the Ritz method, i.e., the first term on the right-hand side of (12.28), is indeed positive definite.

   Hint. Apply the Friedrichs inequality (12.25) to $w_N(x,y) = \sum_{n=1}^{N} c_n w_n^{(N)}$.

2. Consider a Dirichlet problem for the elliptic equation with variable coefficients:

$$\frac{\partial}{\partial x}\left(a(x,y)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(b(x,y)\frac{\partial u}{\partial y}\right) = \varphi(x,y), \quad (x,y) \in \Omega,$$
$$u\Big|_{\Gamma} = \psi(s), \quad \Gamma = \partial\Omega,$$

   where $a(x,y) \geq a_0 > 0$ and $b(x,y) \geq b_0 > 0$. Prove that its solution minimizes the functional:

$$I(w) = \iint\limits_{\Omega} \left[a\left(\frac{\partial u}{\partial x}\right)^2 + b\left(\frac{\partial u}{\partial y}\right)^2 + 2\varphi w\right] dxdy$$

   on the set of all functions $w \in W$ that satisfy the boundary condition: $w\big|_{\Gamma} = \psi(s)$.

3. Let $\psi(s) \equiv 0$ in the boundary value problem of Exercise 2. Apply the Ritz method and obtain the corresponding system of linear algebraic equations.

4. Let $\psi(s) \equiv 0$ in the boundary value problem of Exercise 2. Apply the Galerkin method and obtain the corresponding system of linear algebraic equations.

5. Let the geometry of the two neighboring triangles that have common side $\left[Z_m^{(N)}, Z_n^{(N)}\right]$ be known. Evaluate the coefficients (12.39a) and (12.39b) for the finite element systems (12.37) and (12.38).

6. Consider problem (12.15) with $\psi(s) \equiv 0$ on a square domain Ω. Introduce a uniform Cartesian grid with square cells on the domain Ω. Partition each cell into two right triangles by the diagonal; in doing so, use the same orientation of the diagonal for all cells. Apply the Ritz method on the resulting triangular grid and show that it is equivalent to the central-difference scheme (12.5), (12.6).