Chapter 12

Discrete Methods for Elliptic Problems

The simplest example of an elliptic partial differential equation is the Poisson equation in two space dimensions:

$$\Delta u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \varphi(x,y). \tag{12.1}$$

Consider a domain $\Omega \in \mathbb{R}^2$ with the boundary $\Gamma = \partial\Omega$ and suppose that the following boundary condition supplements equation (12.1):

$$\left.\left(\alpha u + \beta \frac{\partial u}{\partial n}\right)\right|_{\Gamma} = \psi(s), \tag{12.2}$$

where n denotes the direction of differentiation along the outward normal to Γ, $\alpha \geq 0$ and $\beta \geq 0$ are two fixed numbers, $\alpha^2 + \beta^2 = 1$, and s is the arc length along Γ. The functions $\varphi = \varphi(x,y)$ and $\psi = \psi(s)$ are assumed given. The combination of equations (12.1) and (12.2) is called a boundary value problem. If $\alpha = 1$ and $\beta = 0$, then it is a boundary value problem of the first kind, or Dirichlet boundary value problem. If $\alpha = 0$ and $\beta = 1$, then it is a boundary value problem of the second kind, or Neumann boundary value problem. If $\alpha > 0$ and $\beta > 0$, then it is a boundary value problem of the third kind, or Robin boundary value problem.

The foregoing three boundary value problems are the most frequently encountered problems for the Poisson equation, although other problems for this equation are also analyzed in the literature. Along with the Poisson equation, one often considers elliptic equations with variable coefficients, such as:

$$\frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(b\frac{\partial u}{\partial y}\right) = \varphi(x,y),$$

where $a = a(x,y) > 0$ and $b = b(x,y) > 0$. For this equation, typical formulations of boundary value problems also involve boundary conditions (12.2).

Elliptic equations and systems of elliptic equations often appear when modeling various steady-state (i.e., time-independent) processes and phenomena. For example, the Poisson equation (12.1) may govern an electrostatic potential, the velocity potential for a steady-state flow of inviscid incompressible fluid, the steady-state temperature field in a homogeneous and isotropic material, the distribution of concentration for a substance that undergoes steady-state diffusion, etc.

Closed form analytic solutions for elliptic boundary value problems are only available on rare special occasions. Therefore, for most practical purposes those problems need to be solved numerically. In simple cases, when the solution across the entire domain is expected to vary only gradually, and the domain itself has a regular shape, one can use structured (often uniform) grids and construct finite-difference schemes by replacing the derivatives in the equation with appropriate difference quotients.

Otherwise, a structured grid (uniform or even non-uniform) may become unsuitable. Indeed, the regions of rapid variation of the solution, as well as the "bottleneck" parts of the domain (see, e.g., Figure 2.1 on page 59), will necessitate using a very fine grid. However, a structured grid, such as Cartesian, is difficult to refine locally in a multi-dimensional setting. An alternative is provided by unstructured grids, often obtained by triangulation. However, it is impossible to discretize differential equations on such grids by replacing the derivatives with difference quotients. Instead, one often builds discretizations that employ the variational formulation of the problem. The corresponding large group of numerical methods is commonly referred to as the method of finite elements.

In this chapter, we will address two questions. First, in Section 12.1 we will show that a simple central-difference scheme for the Dirichlet problem on a rectangular domain (we encountered this scheme previously, e.g., in Section 5.1.3) is consistent and stable, and as such, converges when the grid is refined. Then, in Section 12.2 we will provide a very brief and introductory account of the method of finite elements, including variational formulations of boundary value problems, the Ritz and Galerkin approximations, and basic concepts related to convergence.

Prior to actually starting the discussion on the subject, let us note that the discretization of linear elliptic boundary value problems, whether by finite differences or by finite elements, typically leads to large (i.e., high-order) systems of linear algebraic equations. Systems of this type can be solved either by direct methods analyzed in Chapter 5 of the book (Gaussian elimination including its tri-diagonal/banded versions, Cholesky factorization, separation of variables and FFT), or by iterative methods analyzed in Chapter 6 of the book (Richardson, Chebyshev, conjugate gradients, Krylov subspace methods — all normally with preconditioning, multigrid).

12.1 A Simple Finite-Difference Scheme. The Maximum Principle

Consider a Dirichlet problem for the Poisson equation on the square $\Omega = \{(x,y) \,\big|\, 0<x<1,\ 0<y<1\}$ with the boundary $\Gamma = \partial\Omega$:

$$\begin{gathered} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \varphi(x,y), \quad (x,y)\in\Omega, \\ u\Big|_{\Gamma} = \psi(s). \end{gathered} \tag{12.3}$$