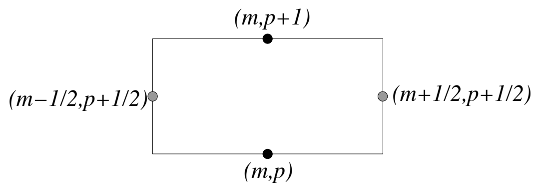

FIGURE 11.9: Grid cell.

Consider two families of straight lines on the plane (x,t): the horizontal lines $t = p\tau$, $p = 0,1,2,\ldots$, and the vertical lines $x = (m+1/2)h$, $m = 0,\pm 1,\pm 2,\ldots$. These lines partition the plane into rectangular cells. On the sides of each cell we will mark the respective midpoints, see Figure 11.9, and compose the overall grid D_h of the resulting nodes (we are not showing the coordinate axes in Figure 11.9).

The unknown function $[u]_h$ will be defined on the grid D_h. Unlike in many previous examples, when $[u]_h$ was introduced as a mere trace of the continuous exact solution $u(x,t)$ on the grid, here we define $[u]_h$ by averaging the solution $u(x,t)$ over the side of the grid cell (see Figure 11.9) that the given node belongs to:

$$[u]_h\Big|_{(x_m,t_p)} \stackrel{\text{def}}{=} \tilde{u}_m^p = \frac{1}{h}\int\limits_{x_{m-1/2}}^{x_{m+1/2}} u(x,t_p)dx,$$

$$[u]_h\Big|_{(x_{m+1/2},t_{p+1/2})} \stackrel{\text{def}}{=} \tilde{U}_{m+1/2}^{p+1/2} = \frac{1}{\tau}\int\limits_{t_p}^{t_{p+1}} u(x_{m+1/2},t)dt.$$

The approximate solution $u^{(h)}$ of our problem will be defined on the same grid D_h. The values of $u^{(h)}$ at the nodes (x_m,t_p) of the grid that belong to the horizontal sides of the rectangles, see Figure 11.9, will be denoted by u_m^p, and the values of the solution at the nodes $(x_{m+1/2},t_{p+1/2})$ that belong to the vertical sides of the rectangles will be denoted by $U_{m+1/2}^{p+1/2}$.

Instead of the discrete function $u^{(h)}$ defined only at the grid nodes (m,p) and $(m+1/2,p+1/2)$, let us consider a family of piecewise constant functions of continuous argument defined on the horizontal and vertical lines of the grid. In other words, we will think of the value u_m^p as associated with the entire horizontal side $\{(x,t)|x_{m-1/2} < x < x_{m+1/2},\ t = t_p\}$ of the grid cell that the node (x_m,t_p) belongs to, see Figure 11.9. Likewise, the value $U_{m+1/2}^{p+1/2}$ will be defined on the entire vertical grid interval $\{(x,t)|x = x_{m+1/2},\ t_p < t < t_{p+1}\}$. The relation between the quantities u_m^p and $U_{m+1/2}^{p+1/2}$, where $m = 0,\pm 1,\pm 2,\ldots$ and $p = 0,1,2,\ldots$, will be established based on the integral conservation law (11.1a) for $k = 1$:

$$\oint\limits_{\Gamma} \frac{u^2}{2}dt - udx = 0.$$

Let us consider the boundary of the grid cell from Figure 11.9 as the contour Γ:

$$\oint_{\Gamma} \frac{(u^{(h)})^2}{2} dt - u^{(h)} dx = 0. \tag{11.16}$$

Using the actual values of the foregoing piecewise constant function $u^{(h)}$, we can rewrite equality (11.16) as follows:

$$h[u_m^{p+1} - u_m^p] + \frac{\tau}{2}\left[\left(U_{m+1/2}^{p+1/2}\right)^2 - \left(U_{m-1/2}^{p+1/2}\right)^2\right] = 0. \tag{11.17}$$

Formula (11.17) implies that if there was a rule for evaluation of the quantities $\left(U_{m+1/2}^{p+1/2}\right)^2$, $m = 0, \pm 1, \pm 2, \ldots$, given the quantities u_m^p, $m = 0, \pm 1, \pm 2, \ldots$, then we could have advanced one time step and obtained u_m^{p+1}, $m = 0, \pm 1, \pm 2, \ldots$. In other words, formula (11.17) would have enabled a marching algorithm. Note that the quantities $\left(U_{m+1/2}^{p+1/2}\right)^2$ are commonly referred to as fluxes. The reason is that the Burgers equation can be equivalently recast in the divergence form:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = \frac{\partial u}{\partial t} + \frac{\partial F(u)}{\partial x} = 0,$$

where $F(u)$ is known as the flux function for the general equation: $u_t + F_x(u) = 0$.

The fluxes $\left(U_{m+1/2}^{p+1/2}\right)^2$ can be computed using various approaches. However, regardless of the specific approach, the finite difference scheme (11.17) always appears conservative. This important characterization means the following.

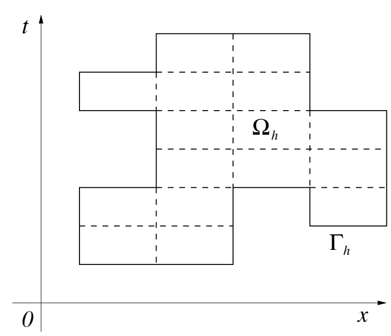

FIGURE 11.10: Grid domain.

Let us draw an arbitrary non-self-intersecting closed contour in the upper semi-plane $t > 0$ that would be completely composed of the grid segments, see Figure 11.10. Accordingly, this contour Γ_h encloses some domain Ω_h composed of the grid cells. Next, let us perform termwise summation of all the equations (11.17) that correspond to the grid cells from the domain Ω_h. Since equations (11.17) and (11.16) are equivalent and the only difference is in the notations, we may think that the summation is performed on equations (11.16). This immediately yields:

$$\oint_{\Gamma_h} \frac{(u^{(h)})^2}{2} dt - u^{(h)} dx = 0. \tag{11.18}$$

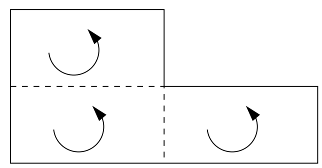

FIGURE 11.11: Directions of integration.

Formula (11.18) is easy to justify. The integrals along those sides of the grid rectangles that do not belong to the boundary Γ_h of the domain Ω_h, see Figure 11.10, mutually cancel out. Indeed, each of these interior grid segments belongs to two neighboring cells. Consequently, the integration of the function $u^{(h)}$ along each of those appears twice in the sum (11.18) and is conducted in the opposite directions, see Figure 11.11. Hence only the contributions due to the exterior boundary Γ_h do not cancel, and we arrive at equality (11.18).

Scheme (11.17) provides an example of what is known as conservative finite-difference schemes. In general, given a scheme, if we perform termwise summation of its finite-difference equations over the nodes of the grid domain Ω_h, and only those contributions to the sum remain that correspond to the boundary Γ_h, then the scheme is called conservative. Conservative schemes are analogous to the differential equations of divergence type, for example:

$$\operatorname{div}\phi = \frac{\partial \phi_1}{\partial t} + \frac{\partial \phi_2}{\partial x} = 0.$$

Once integrated over a two-dimensional domain Ω, these equations give rise to a contour integral along $\Gamma = \partial\Omega$, see formula (11.3). Finite-difference scheme (11.14) is not conservative, whereas scheme (11.17) is conservative.

REMARK 11.1 Let the grid function $u^{(h)}$ that satisfies equation (11.17) for $m = 0, \pm 1, \pm 2, \ldots$ and $p = 0, 1, 2, \ldots$, converge to a piecewise continuous function $u(x,t)$ when $h \longrightarrow 0$ uniformly on any closed region of space that does not contain the discontinuities. Also let $u^{(h)}$ be uniformly bounded with respect to h. Then, $u(x,t)$ satisfies the integral conservation law:

$$\oint_{\Gamma} \frac{u^2}{2} dt - u dx = 0,$$

where Γ is an arbitrary piecewise smooth contour. In other words, $u^{(h)}$ converges to the generalized solution of problem (11.2). This immediately follows from the possibility to approximate Γ by Γ_h, formula (11.18), and the convergence that we have just assumed. ▯

For the difference scheme (11.17) to make sense, we still need to define a procedure for evaluating the fluxes $\left(U_{m+1/2}^{p+1/2}\right)^2$ given the quantities u_m^p. To do that, we can exploit the solution to a special Riemann problem. This approach leads to one of the most popular and successful conservative schemes known as the Godunov scheme.