respectively, see Figure 11.4. It turns out that the values of $u_{\text{left}}(x,t)$ and $u_{\text{right}}(x,t)$ are related to the velocity of the jump $\dot{x} = dx/dt$ in a particular way, and altogether these quantities are not independent.

Let us introduce a contour $ABCD$ that straddles a part of the trajectory of the jump L, see Figure 11.4. The integral conservation law (11.1a) holds for any closed contour Γ and in particular for the contour $ABCDA$:

$$\int\limits_{ABCDA} \frac{u^{k+1}}{k+1}dt - \frac{u^k}{k}dx = 0. \tag{11.8}$$

Next, we start contracting this contour toward the curve L, i.e., start making it narrower. In doing so, the intervals BC and DA will shrink toward the points E and F, respectively, and the corresponding contributions to the integral (11.8) will obviously approach zero, so that in the limit we obtain:

$$\int\limits_{L'} \left[\frac{u^{k+1}}{k+1}\right] dt - \left[\frac{u^k}{k}\right] dx = 0,$$

or alternatively:

$$\int\limits_{L'} \left(\left[\frac{u^{k+1}}{k+1}\right] - \left[\frac{u^k}{k}\right] \frac{dx}{dt} \right) dt = 0.$$

Here the rectangular brackets: $[z] \stackrel{\text{def}}{=} z_{\text{right}} - z_{\text{left}}$ denote the magnitude of the jump of a given quantity z across the discontinuity, and L' denotes an arbitrary stretch of the jump trajectory.

Since L' is arbitrary, the integrand in the previous equality must be equal to zero at every point:

$$\left(\left[\frac{u^{k+1}}{k+1}\right] - \left[\frac{u^k}{k}\right] \frac{dx}{dt} \right) \Bigg|_{(x,t)\in L} = 0,$$

and consequently,

$$\frac{dx}{dt} = \left[\frac{u^{k+1}}{k+1}\right] \cdot \left[\frac{u^k}{k}\right]^{-1}. \tag{11.9}$$

Formula (11.9) indicates that for different values of k we can obtain different conditions at the trajectory of discontinuity L. For example, if $k = 1$ we have:

$$\frac{dx}{dt} = \frac{u_{\text{left}} + u_{\text{right}}}{2}, \tag{11.10}$$

and if $k = 2$ we can write:

$$\frac{dx}{dt} = \frac{2}{3} \frac{u_{\text{left}}^2 + u_{\text{left}} u_{\text{right}} + u_{\text{right}}^2}{u_{\text{left}} + u_{\text{right}}}.$$

We therefore conclude that the conditions that any discontinuous solution of problem (11.1a), (11.1b) must satisfy at the jump trajectory L depend on k.

11.1.4 Generalized Solution of a Differential Problem

Let us define a generalized solution to problem (11.2). This solution can be discontinuous, and we simply identify it with the solution to the integral conservation law (11.1a), (11.1b). Often the generalized solution is also called a weak solution.

In the case of a solution that has continuous derivatives everywhere, we have seen (Section 11.1.1) that the weak solution, i.e., solution to problem (11.1a), (11.1b), does not depend on k and coincides with the classical solution to the Cauchy problem (11.2). In other words, the solution in this case is a differentiable function $u = u(x,t)$ that turns the Burgers equation $u_t + uu_x = 0$ into an identity and also satisfies the initial condition $u(x,0) = \psi(x)$. We have also seen that even in the continuous case it is very helpful to consider both the integral formulation (11.1a), (11.1b) and the differential Cauchy problem (11.2). By staying only within the framework of problem (11.1a), (11.1b), we would make it more difficult to reveal the mechanism of the formation of discontinuity, as done in Section 11.1.2.

In the discontinuous case, the definition of a weak solution to problem (11.2) that we have just introduced does not enhance the formulation of problem (11.1a), (11.1b) yet; it merely renames it. Let us therefore provide an alternative definition of the generalized solution to problem (11.2). In doing so, we will only consider bounded solutions to problem (11.1a), (11.1b) that have continuous first partial derivatives everywhere on the strip $0 < t < T$, except perhaps for a set of smooth curves $x = x(t)$ along which the solution may undergo discontinuities of the first kind (jumps).

DEFINITION 11.1 *The function $u = u(x,t)$ is called a generalized (weak) solution to the Cauchy problem (11.2) that corresponds to the integral conservation law (11.1a) if:*

1. *The function $u = u(x,t)$ satisfies the Burgers equation [see formula (11.2)] at every point of the strip $0 < t < T$ that does not belong to the curves $x = x(t)$ which define the jump trajectories.*
2. *Condition (11.9) holds at the jump trajectory.*
3. *For every x for which the initial function $\psi = \psi(x)$ is continuous, the solution $u = u(x,t)$ is continuous at the point $(x,0)$ and satisfies the initial condition $u(x,0) = \psi(x)$.*

The proof of the equivalence of Definition 11.1 and the definition of a generalized solution given in the beginning of this section is the subject of Exercise 1.

Let us emphasize that in the discontinuous case, the generalized solution of the Cauchy problem (11.2) is not determined solely by equalities (11.2) themselves. It also requires that a particular conservation law, i.e., particular value of k, be specified that would relate the jump velocity with the magnitude of the jump across the discontinuity, see formula (11.9). Note that the general formulation of problem (11.1a), (11.1b) we have adopted provides no motivation for selecting any preferred value of k. However, in the problems that originate from real-world scientific applications, the integral conservation laws analogous to (11.1a) would normally express the conservation of some actual physical quantities. These conservation laws are, of course, well defined. In our subsequent considerations, we will assume for definiteness that the integral conservation law (11.1a) that corresponds to the value of $k = 1$ holds. Accordingly, condition (11.10) is satisfied at the jump trajectory.

In the literature, the pioneering work on weak solutions was done by Lax [Lax54].

11.1.5 The Riemann Problem

Having defined weak solutions of problem (11.2), see Definition 11.1, we will now see how a given initial discontinuity evolves when governed by the Burgers equation. In the literature, the problem of evolution of a discontinuity specified in the initial data is known as the Riemann problem.

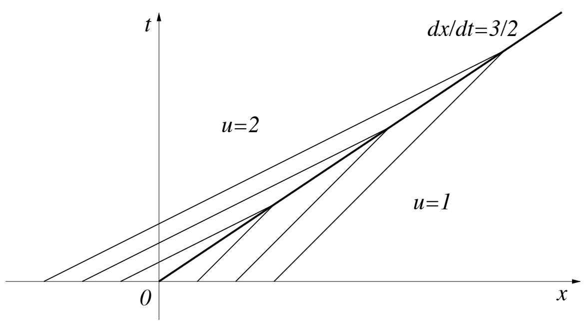


FIGURE 11.5: Shock.

Consider problem (11.2) with the following discontinuous initial function:

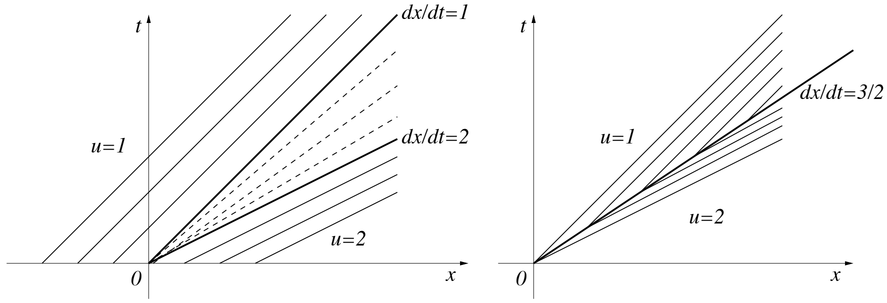
$$\psi(x) = \begin{cases} 2, & x < 0, \\ 1, & x > 0. \end{cases}$$

The corresponding solution is shown in Figure 11.5. The evolution of the initial discontinuity consists of its propagation with the speed $\dot{x} = (2+1)/2 = 3/2$. This speed, which determines the slope of the jump trajectory in Figure 11.5, is obtained according to formula (11.10) as the arithmetic mean of the slopes of characteristics to the left and to the right of the shock. As can be seen, the characteristics on either side of the discontinuity impinge on it. In this case the discontinuity is called a shock; it is similar to the shock waves in the flows of ideal compressible fluid. One can show that the solution from Figure 11.5 is stable with respect to small perturbations of the initial data.

Next consider a different type of initial discontinuity:

$$\psi(x) = \begin{cases} 1, & x < 0, \\ 2, & x > 0. \end{cases} \tag{11.11}$$

One can obtain two alternative solutions for the initial data (11.11). The solution shown in Figure 11.6(a) has no discontinuities for $t > 0$. It consists of two regions with $u = 1$ and $u = 2$ bounded by the straight lines $\dot{x} = 1$ and $\dot{x} = 2$, respectively, that originate from $(0,0)$. These lines do not correspond to the trajectories of discontinuities, as the solution is continuous across both of them. The region in between these two lines is characterized by a family of characteristics that all originate at the



(a) Fan. (b) Unstable discontinuity.

FIGURE 11.6: Solutions of the Burgers equation with initial data (11.11).

same point $(0,0)$. This structure is often referred to as a fan (of characteristics); in the context of gas dynamics it is known as the rarefaction wave.

The solution shown in Figure 11.6(b) is discontinuous; it consists of two regions $u = 1$ and $u = 2$ separated by the discontinuity with the same trajectory $\dot{x} = (1+2)/2 = 3/2$ as shown in Figure 11.5. However, unlike in the case of Figure 11.5, the characteristics in Figure 11.6(b) emanate from the discontinuity and veer away as the time elapses rather than impinge on it; this discontinuity is not a shock.

To find out which of the two solutions is actually realized, we need to incorporate additional considerations. Let us perturb the initial function $\psi(x)$ of (11.11) and consider:

$$\psi(x) = \begin{cases} 1, & x < 0, \\ 1 + x/\varepsilon, & 0 \le x \le \varepsilon, \\ 2, & x > \varepsilon. \end{cases} \tag{11.12}$$

The function $\psi(x)$ of (11.12) is continuous, and the corresponding solution $u(x,t)$ of problem (11.2) is determined uniquely. It is shown in Figure 11.7. When ε tends to zero, this solution approaches the continuous fan-type solution of problem (11.2), (11.11) shown in Figure 11.6(a). At the same time, the discontinuous solution of problem (11.2), (11.11) shown in Figure 11.6(b) appears unstable with respect to small perturbations of the initial data. Hence, it is the continuous solution with the fan that should

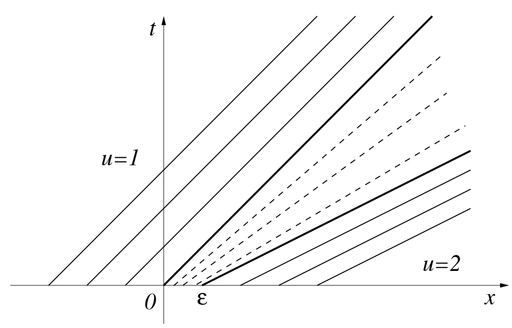


FIGURE 11.7: Solution of the Burgers equation with initial data (11.12).

be selected as the true solution of problem (11.2), (11.11), see Figure 11.6(a). As for the discontinuous solution from Figure 11.6(b), its exclusion due to the instability is similar to the exclusion of the so-called rarefaction shocks that appear as mathematical artifacts when analyzing the flows of ideal compressible fluid. Unstable solutions of this type are prohibited by the so-called entropy conditions that are introduced and analyzed in the theory of quasi-linear hyperbolic equations, see, e.g., [RJ83].

Exercises

1. Prove that the definition of a weak solution to problem (11.2) given in the very beginning of Section 11.1.4 is equivalent to Definition 11.1.

2.* Consider the following auxiliary problem:

$$\begin{gathered}\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\mu\frac{\partial^2 u}{\partial x^2}, \quad 0<t<T, \quad -\infty<x<\infty, \\ u(x,0)=\psi(x), \quad -\infty<x<\infty,\end{gathered} \tag{11.13}$$

where $\mu>0$ is a parameter which is similar to viscosity in the context of fluid dynamics. The differential equation of (11.13) is parabolic rather than hyperbolic. It is known to have a smooth solution for any smooth initial function $\psi(x)$. If the initial function is discontinuous, the solution is also known to become smoother as time elapses.

Let $\psi(x)=2$ for $x<0$ and $\psi(x)=1$ for $x>0$. Prove that when $\mu\longrightarrow 0$, the solution of problem (11.13) approaches the generalized solution of problem (11.2) (see Definition 11.1) that corresponds to the conservation law (11.1a) with $k=1$.

Hint. The solution $u=u(x,t)$ of problem (11.13) can be calculated explicitly with the help of the Hopf formula:

$$u(x,t)=\frac{\int\limits_{-\infty}^{\infty}(x-\xi)e^{-\frac{\lambda(x,\xi,t)}{2\mu}}d\xi}{\int\limits_{-\infty}^{\infty}te^{-\frac{\lambda(x,\xi,t)}{2\mu}}d\xi}, \quad \text{where} \quad \lambda(x,\xi,t)=\frac{(x-\xi)^2}{2t}+\int\limits_0^{\xi}\psi(\eta)d\eta.$$

More detail can be found in the monograph [RJ83], as well as in the original research papers [Hop50] and [Col51].

11.2 Construction of Difference Schemes

In this section, we will provide examples of finite-difference schemes for computing the generalized solution (see Definition 11.1) of problem (11.2):

$$\begin{gathered}\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=0, \quad 0<t<T, \quad -\infty<x<\infty, \\ u(x,0)=\psi(x), \quad -\infty<x<\infty,\end{gathered}$$

that corresponds to the integral conservation law (11.1a) with $k=1$.