

that satisfies the integral equation:

$$\int_{\Gamma} \frac{u^{k+1}}{k+1} dt - \frac{u^k}{k} dx = 0 \quad (11.1a)$$

for an arbitrary closed contour Γ . The quantity k in formula (11.1a) is a fixed positive integer. We also require that $u = u(x, t)$ satisfies the initial condition:

$$u(x, 0) = \psi(x), \quad -\infty < x < \infty. \quad (11.1b)$$

The left-hand side of equation (11.1a) can be interpreted as the flux of the vector field:

$$\phi(x, t) \equiv \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} u^k/k \\ u^{k+1}/(k+1) \end{bmatrix}$$

through the contour Γ . The requirement that the flux of this vector field through an arbitrary contour Γ be equal to zero can be thought of as a conservation law written in an integral form.

Problem (11.1a), (11.1b) provides the simplest formulation that leads to the formation of discontinuities albeit smooth initial data. It can serve as a model for understanding the methods of solving similar problems in the context of fluid dynamics.

11.1 Differential Form of an Integral Conservation Law

11.1.1 Differential Equation in the Case of Smooth Solutions

Let us first assume that the solution $u = u(x, t)$ to problem (11.1a), (11.1b) is continuously differentiable everywhere on the strip $0 \leq t \leq T$. We will then show that problem (11.1a), (11.1b) is equivalent to the following Cauchy problem:

$$\begin{aligned} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} &= 0, \quad 0 < t < T, \quad -\infty < x < \infty, \\ u(x, 0) &= \psi(x), \quad -\infty < x < \infty. \end{aligned} \quad (11.2)$$

In the literature, the differential equation of (11.2) is known as the Burgers equation.

To establish the equivalence of problem (11.1a), (11.1b) and problem (11.2), we recall Green's formula. Let Ω be an arbitrary domain on the (x, t) plane, let $\Gamma = \partial\Omega$ be its boundary, and let the functions $\phi_1(x, t)$ and $\phi_2(x, t)$ have partial derivatives with respect to x and t on the domain Ω that are continuous everywhere up to the boundary Γ . Then, the following Green's formula holds:

$$\iint_{\Omega} \left(\frac{\partial \phi_1}{\partial t} + \frac{\partial \phi_2}{\partial x} \right) dx dt = \int_{\Gamma} \phi_2 dt - \phi_1 dx. \quad (11.3)$$

Identity (11.3) means that the integral of the divergence $\frac{\partial \phi_1}{\partial t} + \frac{\partial \phi_2}{\partial x}$ of the vector field $\phi = [\phi_1, \phi_2]^T$ over the domain Ω is equal to the flux of this vector field through the boundary $\Gamma = \partial\Omega$.

Using formula (11.3), we can write:

$$\int_{\Gamma} \frac{u^{k+1}}{k+1} dt - \frac{u^k}{k} dx = \iint_{\Omega} \left[\frac{\partial}{\partial t} \left(\frac{u^k}{k} \right) + \frac{\partial}{\partial x} \left(\frac{u^{k+1}}{k+1} \right) \right] dx dt. \quad (11.4)$$

Equality (11.4) implies that if a smooth function $u = u(x, t)$ satisfies the Burgers equation, see formula (11.2), then equation (11.1a) also holds. Indeed, if the Burgers equation is satisfied, then we also have:

$$u^{k-1} \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \right) \equiv \frac{\partial}{\partial t} \left(\frac{u^k}{k} \right) + \frac{\partial}{\partial x} \left(\frac{u^{k+1}}{k+1} \right) = 0. \quad (11.5)$$

Consequently, the right-hand side of equality (11.4) becomes zero. The converse is also true: If a smooth function $u = u(x, t)$ satisfies the integral conservation law (11.1a), then at every point (\tilde{x}, \tilde{t}) of the strip $0 < t < T$ equation (11.5) holds, and hence equation (11.2) is true as well. To justify that, let us assume the opposite, and let us, for definiteness, take some point (\tilde{x}, \tilde{t}) for which:

$$\frac{\partial}{\partial t} \left(\frac{u^k}{k} \right) + \frac{\partial}{\partial x} \left(\frac{u^{k+1}}{k+1} \right) \Big|_{(\tilde{x}, \tilde{t})} > 0.$$

Then, by continuity, we can always find a sufficiently small disk $\Omega \subset \{(x, t) | 0 < t < T\}$ centered at (\tilde{x}, \tilde{t}) such that

$$\frac{\partial}{\partial t} \left(\frac{u^k}{k} \right) + \frac{\partial}{\partial x} \left(\frac{u^{k+1}}{k+1} \right) \Big|_{(x, t) \in \Omega} > 0.$$

Hence, combining equations (11.1a) and (11.4) we obtain (recall, $\Gamma = \partial\Omega$):

$$0 = \int_{\Gamma} \frac{u^{k+1}}{k+1} dt - \frac{u^k}{k} dx = \iint_{\Omega} \left[\frac{\partial}{\partial t} \left(\frac{u^k}{k} \right) + \frac{\partial}{\partial x} \left(\frac{u^{k+1}}{k+1} \right) \right] dx dt > 0.$$

The contradiction we have just arrived at, $0 > 0$, proves that for smooth functions $u = u(x, t)$, problem (11.1a), (11.1b) and the Cauchy problem (11.2) are equivalent.

11.1.2 The Mechanism of Formation of Discontinuities

Let us first suppose that the solution $u = u(x, t)$ of the Cauchy problem (11.2) is smooth. Then we introduce the curves $x = x(t)$ defined by the following ordinary differential equation:

$$\frac{dx}{dt} = u(x, t). \quad (11.6)$$