Then, taking into account that $u_0^{p+1} = \vartheta(t_{p+1})$ and $u_M^{p+1} = \chi(t_{p+1})$, we obtain the maximum principle (10.205).

Let us now split the solution $u^{(h)}$ of the problem $L_h u^{(h)} = f^{(h)}$ into two components: $u^{(h)} = v^{(h)} + w^{(h)}$, where $v^{(h)}$ and $w^{(h)}$ satisfy the equations:

$$\boldsymbol{L}_h v^{(h)} = \begin{cases} 0 \\ \psi(x_m) \\ \vartheta(t_{p+1}) \\ \chi(t_{p+1}) \end{cases} \quad \text{and} \quad \boldsymbol{L}_h w^{(h)} = \begin{cases} \varphi(x_m, t_p) \\ 0 \\ 0 \\ 0 \end{cases}.$$

For the solution of the first sub-problem, the maximum principle (10.205) yields:

$$\begin{aligned} \max_m |v_m^{p+1}| &\leq \max\{\max_p |\vartheta(t_p)|, \max_p |\chi(t_p)|, \max_m |v_m^p|\}, \\ \max_m |v_m^p| &\leq \max\{\max_p |\vartheta(t_p)|, \max_p |\chi(t_p)|, \max_m |v_m^{p-1}|\}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \max_m |v_m^1| &\leq \max\{\max_p |\vartheta(t_p)|, \max_p |\chi(t_p)|, \max_m |\psi(x_m)|\}. \end{aligned}$$

For the solution of the second sub-problem, we obtain by virtue of the same estimate (10.205):

$$\begin{aligned} \max_m |w_m^{p+1}| &\leq \max_m |w_m^p| + \tau_p \max_{m,p} |\varphi(x_m, t_p) \\ &\leq \max_m |w_m^{p-1}| + (\tau_p + \tau_{p-1}) \max_{m,p} |\varphi(x_m, t_p) \\ &\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\leq \underbrace{\max_m |w_m^0|}_{=0} + (\tau_p + \tau_{p-1} + \ldots + \tau_0) \max_{m,p} |\varphi(x_m, t_p) \\ &\leq T \max_{m,p} |\varphi(x_m, t_p)|, \end{aligned}$$

where T is the terminal time, see formula (10.201). From the individual estimates established for v_m^{p+1} and w_m^{p+1} we derive:

$$\begin{aligned} \max_m |u_m^{p+1}| = \max_m |v_m^{p+1} + w_m^{p+1}| &\leq \max_m |v_m^{p+1}| + \max_m |w_m^{p+1}| \\ &\leq \max\{\max_p |\vartheta(t_p)|, \max_p |\chi(t_p)|, \max_m |\psi(x_m)|\} \\ &+ T \max_{m,p} |\varphi(x_m, t_p) \leq c \|f^{(h)}\|_{F_h}, \end{aligned} \tag{10.206}$$

where $c = 2\max\{1, T\}$. Inequality (10.206) holds for all p. Therefore, we can write:

$$\|u^{(h)}\| \leq c \|f^{(h)}\|_{F_h}, \tag{10.207}$$

which implies stability of scheme (10.202) in the sense of Definition 10.2. As such, we have shown that the necessary condition of stability (10.204) given by the principle of frozen coefficients is also sufficient for stability of the scheme (10.202).

Inequality (10.204) indicates that if the heat conduction coefficient $a(x_m,t_p)$ assumes large values near some point $(\tilde{x},\tilde{t})$, then computing the solution at time level $t=t_{p+1}$ will necessitate taking a very small time step $\tau=\tau_p$. Therefore, advancing the solution until a prescribed value of $t=T$ is reached may require an excessively large number of steps, which will make the computation impractical.

Let us also note that the foregoing restriction on the time step is of a purely numerical nature and has nothing to do with the physics behind problem (10.201). Indeed, this problem models the propagation of heat in a spatially one-dimensional structure, e.g., a rod, for which the heat conduction coefficient $a=a(x,t)$ may vary along the rod and also in time. Large values of $a(x,t)$ in a neighborhood of some point $(\tilde{x},\tilde{t})$ merely imply that this neighborhood can be removed, i.e., "cut off," from the rod without changing the overall pattern of heat propagation. In other words, we may think that this part of the rod consists of a material with zero heat capacity.

10.6.2 An Implicit Scheme

Instead of scheme (10.202), we can use the same grid and build the scheme on the stencil shown in Figure 10.3(right) (see page 331):

$$\begin{gathered}
\frac{u_m^{p+1}-u_m^p}{\tau_p}-a(x_m,t_p)\frac{u_{m+1}^{p+1}-2u_m^{p+1}+u_{m-1}^{p+1}}{h^2}=\varphi(x_m,t_{p+1}),\\
m=1,2,\dots,M-1,\\
u_m^0=\psi_m,\quad m=0,1,\dots,M,\\
u_0^{p+1}=\vartheta(t_{p+1}),\quad u_M^{p+1}=\chi(t_{p+1}),\quad p\geq 0,\\
t_0=0,\quad t_p=\tau_0+\tau_1+\dots+\tau_{p-1},\quad p=1,2,\dots.
\end{gathered}\tag{10.208}$$

Assume that the solution u_m^p, $m=0,1,\dots,M$, at the time level $t=t_p$ is already known. According to formula (10.208), in order to compute the values of u_m^{p+1}, $m=0,1,\dots,M$, at the next time level $t=t_{p+1}=t_p+\tau_p$ we need to solve the following system of linear algebraic equations with respect to $u_m\equiv u_m^{p+1}$:

$$\begin{gathered}
u_0=\vartheta(t_{p+1}),\\
\alpha_m u_{m-1}+\beta_m u_m+\gamma_m u_{m+1}=f_m,\quad m=1,2,\dots,M-1,\\
u_M=\chi(t_{p+1}),
\end{gathered}\tag{10.209}$$

where

$$\begin{gathered}
\alpha_m=\gamma_m=-\frac{\tau_p}{h^2}a(x_m,t_p),\quad \beta_m=1+2\frac{\tau_p}{h^2}a(x_m,t_p),\quad f_m=u_m^p+\tau_p\varphi(x_m,t_{p+1}),\\
m=1,2,\dots,M-1,\\
\gamma_0=\alpha_M=0,\quad \beta_0=\beta_M=1.
\end{gathered}$$

It is therefore clear that

$$\begin{gathered}
|\beta_m|=|\alpha_m|+|\gamma_m|+\delta,\quad \delta=1>0,\\
m=0,1,2,\dots,M,
\end{gathered}$$

and because of the diagonal dominance, system (10.209) can be solved by the algorithm of tri-diagonal elimination described in Section 5.4.2. Note that in the case of scheme (10.208) there are no explicit formulae, i.e., closed form expressions such as formula (10.203), that would allow one to obtain the solution u_m^{p+1} at the upper time level given the solution u_m^p at the lower time level. Instead, when marching the solution in time one needs to solve systems (10.209) repeatedly, i.e., on every step, and that is why the scheme (10.208) is called *implicit*.

In Section 10.3.3 (see Example 7), we analyzed an implicit finite-difference scheme for the constant-coefficient heat equation and demonstrated that the von Neumann spectral stability condition holds for this scheme for any value of the ratio $r = \tau/h^2$. By virtue of the principle of frozen coefficients (see Section 10.4.1), the spectral stability condition will not impose any constraints on the time step τ even when the heat conduction coefficient $a(x,t)$ varies. This makes implicit scheme (10.208) unconditionally stable. It can be used efficiently even when the coefficient $a(x,t)$ assumes large values for some $(\tilde{x},\tilde{t})$. For convenience, when computing the solution of problem (10.201) with the help of scheme (10.208), one can choose a constant, rather than variable, time step $\tau_p = \tau$.

To conclude this section, let us note that unconditional stability of the implicit scheme (10.208) can be established rigorously. Namely, one can prove (see [GR87, § 28]) that the solution u_m^p of system (10.208) satisfies the same maximum principle (10.205) as holds for the explicit scheme (10.202). Then, estimate (10.207) for scheme (10.208) can be derived the same way as in Section 10.6.1.

Exercise

1. Let the heat conduction coefficient in problem (10.201) be defined as $a = 1 + u^2$, so that problem (10.201) becomes nonlinear.

   a) Introduce an explicit scheme and an implicit scheme for this new problem.

   b) Consider the following explicit scheme:

$$\frac{u_m^{p+1} - u_m^p}{\tau_p} - [1 + (u_m^p)^2] \frac{u_{m+1}^p - 2u_m^p + u_{m-1}^p}{h^2} = 0,$$
$$m = 1, 2, \dots, M-1,$$
$$u_m^0 = \psi(x_m) \equiv \psi_m, \quad m = 0, 1, \dots, M,$$
$$u_0^{p+1} = \vartheta(t_{p+1}), \quad u_M^{p+1} = \chi(t_{p+1}), \quad p \geq 0,$$
$$t_0 = 0, \quad t_p = \tau_0 + \tau_1 + \dots + \tau_{p-1}, \quad p = 1, 2, \dots.$$

   How should one choose τ_p, given the values of the solution u_m^p at the level p?

   c) Consider an implicit scheme based on the following finite-difference equation:

$$\frac{u_m^{p+1} - u_m^p}{\tau_p} - [1 + (u_m^{p+1})^2] \frac{u_{m+1}^{p+1} - 2u_m^{p+1} + u_{m-1}^{p+1}}{h^2} = 0.$$

   Propose a modification of this equation that would leave the scheme implicit but enable application of the tri-diagonal elimination of Section 5.4 for the transition from u_m^p, $m = 0, 1, \dots, M$, to u_m^{p+1}, $m = 0, 1, \dots, M$?