2.2.5 Saturation of Piecewise Polynomial Interpolation

As we have seen previously (Sections 1.3.1 and 2.2.4), reconstruction of continuous functions from discrete data is normally accompanied by the unavoidable error. This error is caused by the loss of information which inevitably occurs in the course of discretization. The unavoidable error is typically determined by the regularity of the continuous function and by the discretization parameters. Hence, it is not related to any particular reconstruction technique and rather presents a common intrinsic accuracy limit for all such techniques, in particular, for algebraic interpolation.

Therefore, an important question regarding every specific reconstruction method is whether or not it can reach the aforementioned intrinsic accuracy limit. If the accuracy of a given method is limited by its own design and does not, generally speaking, reach the level of the unavoidable error determined by the smoothness of the approximated function, then the method is said to be *saturated by smoothness*. Otherwise, if the accuracy of the method automatically adjusts to the smoothness of the approximated function, then the method does not get saturated.

Let $f(x)$ be defined on the interval $[a,b]$, and let its table of values $f(x_k)$ be known for the equally spaced nodes $x_k = a + kh$, $k = 0,1,\dots,n$, $h = (b-a)/n$. In Section 2.2.2, we saw that the error of piecewise polynomial interpolation of degree s is of order $\mathscr{O}(h^{s+1})$, provided that the polynomial $P_s(x, f_{kj})$ is used to approximate $f(x)$ on the interval $x_k \leq x \leq x_{k+1}$, and that the derivative $f^{(s+1)}(x)$ exists and is bounded. Assume now that the only thing we know about $f(x)$ besides the table of values is that it has a bounded derivative of the maximum order $q+1$. If $q < s$, then the unavoidable error of reconstructing $f(x)$ from its tabulated values is $\mathscr{O}(h^{q+1})$. This is not as good as the $\mathscr{O}(h^{s+1})$ error that the method can potentially achieve, and the reason for deterioration is obviously the lack of smoothness. On the other hand, if $q > s$, then the accuracy of interpolation still remains of order $\mathscr{O}(h^{s+1})$ and does not reach the intrinsic limit $\mathscr{O}(h^{q+1})$. In other words, the order of interpolation error does not react in any way to the additional smoothness of the function $f(x)$, beyond the required $s+1$ derivatives. This is a manifestation of susceptibility of the algebraic piecewise polynomial interpolation to the saturation by smoothness.

In Chapter 3, we discuss an alternative interpolation strategy based on the use of trigonometric polynomials. That type of interpolation appears to be not susceptible to the saturation by smoothness.

Exercises

1. What size of the grid h guarantees that the error of piecewise linear interpolation for the function $f(x) = \sin x$ will never exceed 10^{-6}?

2. What size of the grid h guarantees that the error of piecewise quadratic interpolation for the function $f(x) = \sin x$ will never exceed 10^{-6}?

3. The values of $f(x)$ can be measured at any given point x with the accuracy $|\delta f| \leq 10^{-4}$. Assuming that $|f''(x)| \leq 1$, what is the optimal grid size for tabulating $f(x)$, if the function is to be subsequently reconstructed by means of a piecewise linear interpolation?

Hint. Choosing h excessively small can make the interpolation error smaller than the perturbation of the interpolating polynomial due to the perturbations in the data, see Section 2.1.4.

4.* The same question as in problem 3, but for piecewise quadratic interpolation and assuming that $|f'''(x)| \leq 1$.

5. Consider two approximation formulae for the first derivative $f'(x)$:

$$f'(x) \approx \frac{f(x+h)-f(x)}{h}, \tag{2.40}$$

$$f'(x) \approx \frac{f(x+h)-f(x-h)}{2h}, \tag{2.41}$$

and let $|f''(x)| \leq 1$ and $|f'''(x)| \leq 1$.

a) Find h such that the error of either formula will not exceed 10^{-3}.

b)* Assume that the function f itself is only known with the error δ. What is the best accuracy that one can achieve using formulae (2.40) and (2.41), and how one should properly choose h?

c)* Show that the asymptotic order of the error with respect to δ, obtained by formula (2.41) with the optimal h, cannot be improved.

2.3 Smooth Piecewise Polynomial Interpolation (Splines)

A classical piecewise polynomial interpolation of any degree s, e.g., piecewise linear, piecewise quadratic, etc., see Section 2.2, yields the interpolant that, generally speaking, is not differentiable even once at the interpolation nodes. There are, however, two alternative types of piecewise polynomial interpolants — local and nonlocal splines — that do have a given number of continuous derivatives everywhere, including the interpolation nodes.

2.3.1 Local Interpolation of Smoothness s and Its Properties

Assume that the interpolation nodes x_k and the function values $f(x_k)$ are given. Let us then specify a positive integer number s and also fix another positive integer j: $0 \leq j \leq s-1$. We will associate the interpolating polynomial $P_s(x, f_{kj}) \equiv P_s(x, f, x_{k-j}, x_{k-j+1}, \ldots, x_{k-j+s})$ with every point x_k; this polynomial is built on the nodes $x_{k-j}, x_{k-j+1}, \ldots, x_{k-j+s}$ using the function values $f(x_{k-j}), f(x_{k-j+1}), \ldots, f(x_{k-j+s})$. *A piecewise polynomial local spline* $\varphi(x,s)$ that has continuous derivatives up to the order s is defined individually for each segment $[x_k, x_{k+1}]$ by means of the following equalities:

$$\varphi(x,s) = Q_{2s+1}(x,k), \quad x \in [x_k, x_{k+1}], \quad k = 0, \pm 1, \ldots, \tag{2.42}$$

where each $Q_{2s+1}(x,k)$ is a polynomial of degree no greater than $2s+1$ that satisfies the relations:

$$\left.\frac{d^m Q_{2s+1}(x,k)}{dx^m}\right|_{x=x_k} = \left.\frac{d^m P_s(x,f_{kj})}{dx^m}\right|_{x=x_k}, \qquad m=0,1,2,\ldots,s, \tag{2.43}$$

$$\left.\frac{d^m Q_{2s+1}(x,k)}{dx^m}\right|_{x=x_{k+1}} = \left.\frac{d^m P_s(x,f_{k+1,j})}{dx^m}\right|_{x=x_{k+1}}, \qquad m=0,1,2,\ldots,s. \tag{2.44}$$

THEOREM 2.8
The polynomial $Q_{2s+1}(x,k)$ of degree no greater than $2s+1$ defined by means of equalities (2.43) and (2.44) exists and is unique.

PROOF Let $Q_{2s+1}(x,k) = c_{0,k} + c_{1,k}x + \ldots + c_{2s+1,k}x^{2s+1}$. Then, relations (2.43) and (2.44) together can be considered as a system of $2s+2$ linear algebraic equations with respect to the $2s+2$ unknown coefficients $c_{0,k}$, $c_{1,k}$, ..., $c_{2s+1,k}$. Let us analyze its homogeneous counterpart obtained by replacing all the right-hand sides of equalities (2.43) and (2.44) by zeros. Homogeneity of (2.43), (2.44) clearly implies that the polynomial $Q_{2s+1}(x,k)$ has a root of multiplicity $s+1$ at $x=x_k$ and another root of multiplicity $s+1$ at $x=x_{k+1}$. In other words, $Q_{2s+1}(x,k)$ has a total of $2s+2$ roots counting their multiplicities. This is only possible if $Q_{2s+1}(x,k)\equiv 0$, because $Q_{2s+1}(x,k)$ is a polynomial of degree no greater than $2s+1$. Consequently, $c_{0,k}=c_{1,k}=\ldots=c_{2s+1,k}=0$, and we conclude that the homogeneous counterpart of the linear algebraic system (2.43), (2.44) may only have a trivial solution. As such, the original inhomogeneous system (2.43), (2.44) itself will have a unique solution for any choice of its right-hand sides. $\square$

THEOREM 2.9
Let $f(x)$ be a polynomial of degree no greater than s. Then, the interpolant $\varphi(x,s)$ coincides with this polynomial.

PROOF We will prove the identity $\varphi(x,s)\equiv f(x)$ on the interval $[x_k,x_{k+1}]$ for an arbitrary k, i.e., for all x in between any two neighboring interpolation nodes. In other words, we will prove that $Q_{2s+1}(x,k)\equiv f(x)$. Due to the uniqueness of the interpolating polynomial, we have $P_s(x,f_{kj})\equiv P_s(x,f_{k+1,j})\equiv f(x)$. Then, clearly, the polynomial $f(x)$ solves system (2.43), (2.44). $\square$

THEOREM 2.10
The piecewise polynomial interpolating function $\varphi(x,s)$ defined by equalities (2.42) assumes the given values $f(x_k)$ at the interpolation nodes x_k, $k=0,\pm1,\ldots$. Moreover, $\varphi(x,s)$ has a continuous derivative of order s everywhere on its domain.

PROOF According to equalities (2.43) and (2.44), at any given node x_k the two functions: $Q_{2s+1}(x,k-1)$ and $Q_{2s+1}(x,k)$, have derivatives of orders[3] $m = 0,1,2,\dots,s$ that coincide with the corresponding derivatives of one and the same interpolating polynomial $P_s(x, f_{kj})$. By virtue of equalities (2.42), this proves the theorem. ∎

Let us now recast the polynomial $Q_{2s+1}(x,k)$ as

$$Q_{2s+1}(x,k) = P_s(x, f_{kj}) + R_{2s+1}(x,k), \tag{2.45}$$

where $R_{2s+1}(x,k)$ denotes a correction to the classical interpolating polynomial $P_s(x, f_{kj})$. Then, the following theorem holds.

THEOREM 2.11
The correction $R_{2s+1}(x,k)$ defined by (2.45) can be written in the form:

$$\begin{aligned} &R_{2s+1}(x,k) = \\ &(x_{k+1}-x_k)^{s+1} f(x_{k-j}, x_{k-j+1}, \dots, x_{k-j+s+1})\, q_{2s+1}\left(\frac{x-x_k}{x_{k+1}-x_k}, k\right), \end{aligned} \tag{2.46}$$

where $f(x_{k-j}, x_{k-j+1}, \dots, x_{k-j+s+1})$ is a divided difference of order $s+1$, and

$$\begin{aligned} q_{2s+1}(X,k) = &\left(\frac{x_{k-j+s+1}-x_{k-j}}{x_{k+1}-x_k}\right) \times \\ &\sum_{r=0}^{s} \left\{ \left[\prod_{i=1}^{s} \left(X - \frac{x_{k-j+i}-x_k}{x_{k+1}-x_k} \right) \right]_{X=1}^{(r)} \right\} l_r(X), \qquad X = \frac{x-x_k}{x_{k+1}-x_k}, \end{aligned} \tag{2.47}$$

$$l_r(X) = \frac{X^{s+1}(X-1)^r}{r!s!} \sum_{m=0}^{s-r} (-1)^m \frac{(s+m)!}{m!} (X-1)^m. \tag{2.48}$$

In formula (2.47), expression $[\dots]_{X=1}^{(r)}$ denotes a derivative of order r with respect to X evaluated for $X=1$.

REMARK 2.4 Representation (2.45) of the local piecewise polynomial interpolant with s continuous derivatives can be thought of as Newton's form of the interpolating polynomial $P_{s+1}(x, f_{kj})$:

$$\begin{aligned} P_{s+1}(x, f_{kj}) &= P_s(x, f_{kj}) + f(x_{k-j}, x_{k-j+1}, \dots, x_{k-j+s+1})\phi_{s+1}(x,k), \\ \phi_{s+1}(x,k) &= (x-x_{k-j})(x-x_{k-j+1})\dots(x-x_{k-j+s}), \end{aligned}$$

[3] A derivative of order zero shall naturally be interpreted as the function itself.