Denote $[\|u\|_2']^2 = h\sum\limits_{m=0}^{M-1}|u_m|^2 = \|u\|_2^2 - hu_M^2$. Then the previous inequality implies:

$$[\|u^{p+1}\|_2']^2 \le (1+A\tau)[\|u^p\|_2']^2 + \tau(a_M^p + Ah)\,(\chi^p)^2\,, \quad p = 1,2,\dots,$$

and consequently:

$$[\|u^p\|_2']^2 \le (1+A\tau)^p[\|\psi\|_2']^2 + \sum_{k=1}^{p}(1+A\tau)^{p-k}\tau(a_M^{k-1}+Ah)(\chi^{k-1})^2, \quad p = 1,2,\dots.$$

We again need to distinguish between the cases $A \le 0$, $p = 0,1,2,\dots$, and $A > 0$, $p = 0,1,2,\dots,[T/\tau]$:

$$[\|u^p\|_2']^2 \le \begin{cases} [\|\psi\|_2']^2 + \sum\limits_{k=1}^{\infty}\tau a_M^{k-1}(\chi^{k-1})^2, & A \le 0, \\ e^{AT}\left([\|\psi\|_2']^2 + \sum\limits_{k=1}^{[T/\tau]}\tau(a_M^{k-1}+Ah)(\chi^{k-1})^2\right), & A > 0. \end{cases} \tag{10.182}$$

The discrete estimate (10.182) is analogous to the continuous estimate (10.175). To use the norms $\|\cdot\|_2$ instead of $\|\cdot\|_2'$ in (10.182), we only need to add a bounded quantity $h(\chi^0)^2 + h(\chi^p)^2$ on the right-hand side.

Energy estimates (10.178), (10.180), and (10.182) imply the l_2 stability of the schemes (10.176), (10.179), and (10.181), respectively, in the sense of the Definition 10.2 from page 312. Note that since the foregoing schemes are explicit, stability is not unconditional, and the Courant number has to satisfy $r \le 1$ for scheme (10.176) and $r \le [\sup_{(x,t)} a(x,t)]^{-1}$ for schemes (10.179) and (10.181).

In general, direct energy estimates appear helpful for studying stability of finite-difference schemes. Indeed, they may provide sufficient conditions for those difficult cases that involve variable coefficients, boundary conditions, and even multiple space dimensions. In addition to the scalar hyperbolic equations, energy estimates can be obtained for some hyperbolic systems, as well as for the parabolic equations. For detail, we refer the reader to [GKO95, Chapters 9 & 11], and to some fairly recent journal publications [Str94, Ols95a, Ols95b]. However, there is a key non-trivial step in proving energy estimates for finite-difference initial boundary value problems, namely, obtaining the discrete summation by parts rules appropriate for a given discretization [see the example given by formula (10.177)]. Sometimes, this step may not be obvious at all; otherwise, it may require using alternative norms based on specially chosen inner products.

10.5.4 A Necessary and Sufficient Condition of Stability. The Kreiss Criterion

In Section 10.5.2, we have shown that for stability of a finite-difference initial boundary value problem it is necessary that the spectrum of the family of transition operators $\boldsymbol{R}_h$ belongs to the unit disk on the complex plane. We have also shown, see Theorem 10.9, that this condition is, in fact, not very far from a sufficient one, as it guarantees the scheme from developing a catastrophic exponential instability. However, it is not a fully sufficient condition, and there are examples of the schemes that satisfy the Godunov and Ryaben'kii criterion of Section 10.5.2, i.e., that have their spectrum of $\{\boldsymbol{R}_h\}$ inside the unit disk, yet they are unstable.

A comprehensive analysis of the necessary and sufficient conditions of stability for the schemes that approximate time-dependent problems on finite intervals is rather involved. In the literature, the corresponding series of results is commonly referred to as the Gustafsson, Kreiss, and Sundström (GKS) theory, and we refer the reader to the monograph [GKO95, Part II] for detail. A concise narrative of this theory can also be found in [Str04, Chapter 11]. All results of the GKS theory are formulated in terms of the l_2 norm. An important tool used for obtaining stability estimates is the Laplace transform in time.

Although a full account of (and even a self-contained introduction to) the GKS theory is beyond the scope of this text, its key ideas are easy to understand on the qualitative level and easy to illustrate with examples. The following material is essentially based on that of Section 10.5.2 and can be skipped during the first reading.

Let us consider an initial boundary value problem for the first order constant coefficient hyperbolic equation:

$$\begin{gathered}\frac{\partial u}{\partial t}-\frac{\partial u}{\partial x}=0, \quad 0\le x\le 1, \quad 0<t\le T,\\ u(x,0)=\psi(x), \quad u(1,t)=0.\end{gathered} \tag{10.183}$$

We introduce a uniform grid: $x_m = mh$, $m = 0,1,\dots,M$, $h = 1/M$; $t_p = p\tau$, $p = 0,1,2,\dots$, and approximate problem (10.183) with the leap-frog scheme:

$$\begin{gathered}\frac{u_m^{p+1}-u_m^{p-1}}{2\tau}-\frac{u_{m+1}^{p}-u_{m-1}^{p}}{2h}=0,\\ m=1,2,\dots,M-1, \quad p=1,2,\dots,[T/\tau]-1,\\ u_m^0=\psi(x_m), \quad u_m^1=\psi(x_m+\tau), \quad m=0,1,\dots,M,\\ \boldsymbol{l}u_0^{p+1}=0, \quad u_M^{p+1}=0, \quad p=1,2,\dots,[T/\tau]-1.\end{gathered} \tag{10.184}$$

Notice that scheme (10.184) requires two initial conditions, and for simplicity we use the exact solution, which is readily available in this case, to specify u_m^1 for $m = 0,1,\dots,M-1$. Also notice that the differential problem (10.183) does not require any boundary conditions at the "outflow" boundary $x = 0$, but the discrete problem (10.184) requires an additional boundary condition that we symbolically denote $\boldsymbol{l}u_0^{p+1} = 0$. We will investigate two different outflow conditions for scheme (10.184):

$$u_0^{p+1}=u_1^{p+1} \tag{10.185a}$$

and

$$u_0^{p+1}=u_0^{p}+r(u_1^{p}-u_0^{p}), \tag{10.185b}$$

where we have used our standard notation $r = \frac{\tau}{h} = \text{const}$.

Let us first note that scheme (10.184) is not a one-step scheme, which, in particular, renders the corresponding finite-difference Cauchy problem (mildly) unstable for $r = 1$, see Section 10.3.6. To reduce scheme (10.184) to the canonical form (10.141) so that to be able to investigate the spectrum of the family of operators $\{\boldsymbol{R}_h\}$, we would formally need to introduce additional variables (i.e., transform a scalar equation into a system) and then consider a one-step finite-difference equation, but with vector unknowns. However, it turns out that in this case the Babenko-Gelfand procedure of Section 10.5.1 applied to the resulting vector scheme is equivalent to the Babenko-Gelfand procedure applied directly to the scalar multi-step scheme (10.184). As such, we will skip the formal reduction of scheme (10.184) to the canonical form (10.141) and proceed immediately to computing the spectrum of the corresponding family of transition operators.

We need to analyze three model problems that follow from (10.184): A problem with no lateral boundaries:

$$\begin{gathered}\frac{u_m^{p+1}-u_m^{p-1}}{2\tau}-\frac{u_{m+1}^p-u_{m-1}^p}{2h}=0,\\ m=0,\pm1,\pm2,\dots,\end{gathered}\tag{10.186}$$

a problem with only the left boundary:

$$\begin{gathered}\frac{u_m^{p+1}-u_m^{p-1}}{2\tau}-\frac{u_{m+1}^p-u_{m-1}^p}{2h}=0,\\ m=1,2,\dots,\\ \boldsymbol{l}u_0^{p+1}=0,\end{gathered}\tag{10.187}$$

and a problem with only the right boundary:

$$\begin{gathered}\frac{u_m^{p+1}-u_m^{p-1}}{2\tau}-\frac{u_{m+1}^p-u_{m-1}^p}{2h}=0,\\ m=M-1,M-2,\dots,1,0,-1,\dots,\\ u_M^{p+1}=0.\end{gathered}\tag{10.188}$$

Substituting a solution of the type:

$$u_m^p=\lambda^p u_m$$

into the finite-difference equation:

$$u_m^{p+1}-u_m^{p-1}=r(u_{m+1}^p-u_{m-1}^p),\quad r=\tau/h,$$

that corresponds to all three problems (10.186), (10.187), and (10.188), we obtain the following second order ordinary difference equation for the eigenfunction $\{u_m\}$:

$$(\lambda-\lambda^{-1})u_m-r(u_{m+1}-u_{m-1})=0.\tag{10.189}$$

Its characteristic equation:

$$(\lambda - \lambda^{-1}) - r(q - q^{-1}) = 0 \tag{10.190a}$$

has two roots: $q_1 = q_1(\lambda)$ and $q_2 = q_2(\lambda)$, so that the general solution of equation (10.189) can be written as

$$u_m = c_1 q_1^m + c_2 q_2^m, \quad m = 0, \pm 1, \pm 2, \ldots, \quad c_1 = \text{const}, \quad c_2 = \text{const}.$$

It will also be convenient to recast the characteristic equation (10.190a) in an equivalent form:

$$q^2 - \frac{\lambda - \lambda^{-1}}{r} q - 1 = 0. \tag{10.190b}$$

From equation (10.190b) one can easily see that $q_1 q_2 = -1$ and consequently, unless both roots have unit magnitude, we always have $|q_1(\lambda)| < 1$ and $|q_2(\lambda)| > 1$.

The solution of problem (10.186) must be bounded: $|u_m| \leq \text{const}$ for $m = 0, \pm 1, \pm 2, \ldots$. We therefore require that for this problem $|q_1| = |q_2| = 1$, which means $q_1 = e^{i\alpha}$, $0 \leq \alpha < 2\pi$, and $q_2 = -e^{-i\alpha}$. The spectrum of this problem was calculated in Example 5 of Section 10.3.3:

$$\overleftrightarrow{\Lambda} = \left\{ \lambda(\alpha) = ir\sin\alpha \pm \sqrt{1 - r^2 \sin^2\alpha} \,\middle|\, 0 \leq \alpha < 2\pi \right\}. \tag{10.191}$$

Provided that $r \leq 1$, the spectrum $\overleftrightarrow{\Lambda}$ given by formula (10.191) belongs to the unit circle on the complex plane.

For problem (10.188), we must have $u_m \longrightarrow 0$ as $m \longrightarrow -\infty$. Consequently, its general solution is given by:

$$u_m^p = c_2 \lambda^p q_2^m, \quad m = M, M-1, \ldots, 1, 0, -1, \ldots.$$

The homogeneous boundary condition $u_M^{p+1} = 0$ of (10.184) implies that a nontrivial eigenfunction $u_m = c_2 q_2^m$ may only exist if $\lambda = 0$. From the characteristic equation (10.190a) in yet another equivalent form $(\lambda^2 - 1)q - r\lambda(q^2 - 1) = 0$, we conclude that if $\lambda = 0$ then $q = 0$, which means that problem (10.188) has no eigenvalues:

$$\overleftarrow{\Lambda} = \emptyset. \tag{10.192}$$

To study problem (10.187), we first consider boundary condition (10.185a), known as the extrapolation boundary condition. The solution of problem (10.187) must satisfy $u_m \longrightarrow 0$ as $m \longrightarrow \infty$. Consequently, its general form is:

$$u_m^p = c_1 \lambda^p q_1^m, \quad m = 0, 1, 2, \ldots.$$

The extrapolation condition (10.185a) implies that a nontrivial eigenfunction $u_m = c_1 q_1^m$ may only exist if either $\lambda = 0$ or $c_1(1 - q_1) = 0$. However, we must have $|q_1| < 1$ for problem (10.187), and as such, we see that this problem has no eigenvalues either:

$$\overrightarrow{\Lambda} = \emptyset. \tag{10.193}$$

Combining formulae (10.191), (10.192), and (10.193), we obtain the spectrum of the family of operators:

$$\Lambda = \overleftrightarrow{\Lambda} \cup \overleftarrow{\Lambda} \cup \overrightarrow{\Lambda} = \overleftrightarrow{\Lambda}.$$

We therefore see that according to formula (10.191), the necessary condition for stability (Theorem 10.8) of scheme (10.184), (10.185a) is satisfied when $r \leq 1$.

However, scheme (10.184), (10.185a) still turns out unstable. The instability is not catastrophic, because according to Theorem 10.9, even if there is no uniform bound on the powers of the transition operators, their rate of growth should still be slower than any exponential function. Yet one can clearly see the instability in Figure 10.14, where we show the results of numerical integration of problem (10.183) with $\psi(x) = \cos 2\pi x$ and $u(1,t) = \cos 2\pi(1+t)$ so that $u(x,t) = \cos 2\pi(x+t)$, using scheme (10.184), (10.185a) with $r = 0.95$. (The actual proof of instability can be found, e.g., in [GKO95, Section 13.1] or in [Str04, Section 11.2].) Moreover, as $r < 1$, this instability cannot be attributed to the instability of the finite-difference Cauchy problem for the leap-frog scheme in the case $r = 1$, which is due to a multiple eigenvalue $|\lambda| = 1$, see Section 10.3.6.

In order to analyze what may have caused the instability of scheme (10.184), (10.185a), let us return to the proof of Theorem 10.9. If we were able to claim that the entire spectrum of the family of operators $\{\boldsymbol{R}_h\}$ lies strictly inside the unit disk, then a straightforward modification of that proof would immediately yield a uniform bound on the powers $\boldsymbol{R}_h^p$. This situation, however, is generally impossible. Indeed, in all our previous examples, the spectrum has always contained at least one point on the unit circle: $\lambda = 1$. It is therefore natural to assume that since the points λ inside the unit disk present no danger of instability according to Theorem 10.9, then the potential "culprits" should be sought on the unit circle.

As the finite-difference Cauchy problem (10.186) has no multiple eigenvalues $|\lambda| = 1$ for the case $r < 1$, let us revisit the problem with the left boundary (10.187). We have shown that this problem has no nontrivial eigenfunctions in the class $u_m \longrightarrow 0$ as $m \longrightarrow \infty$ and accordingly, it has no eigenvalues either, see formula (10.193). As such, it does not contribute to the overall spectrum of the family of operators. However, even though the boundary condition (10.185a) in the form $c_1(1-q_1) = 0$ is not satisfied by any function $u_m = c_1 q_1^m$, where $|q_1| < 1$, we see that it is "almost satisfied" if the root q_1 is close to one. Therefore, the function $u_m = c_1 q_1^m$ is "almost an eigenfunction" of problem (10.187), and the smaller the quantity $|1-q_1|$, the more of a genuine eigenfunction it becomes.

To investigate stability, we need to determine whether or not the foregoing "almost an eigenfunction" can bring along an unstable eigenvalue, or rather "almost an eigenvalue," $|\lambda| > 1$. By passing to the limit $q_1 \longrightarrow 1$, we find from equation (10.190a) that $\lambda = 1$ or $\lambda = -1$. We should therefore analyze the behavior of the quantities λ and q in a neighborhood of each of these two values of λ, when the relation between λ and q is given by equation (10.190a).

First recall that according to formula (10.191), if $|q| = 1$, then $|\lambda| = 1$ (provided that $r \leq 1$). Consequently, if $|\lambda| > 1$, then $|q| \neq 1$, i.e., there are two distinct roots: $|q_1| < 1$ and $|q_2| > 1$. In particular, when λ is near $(1,0)$, there are still two roots —

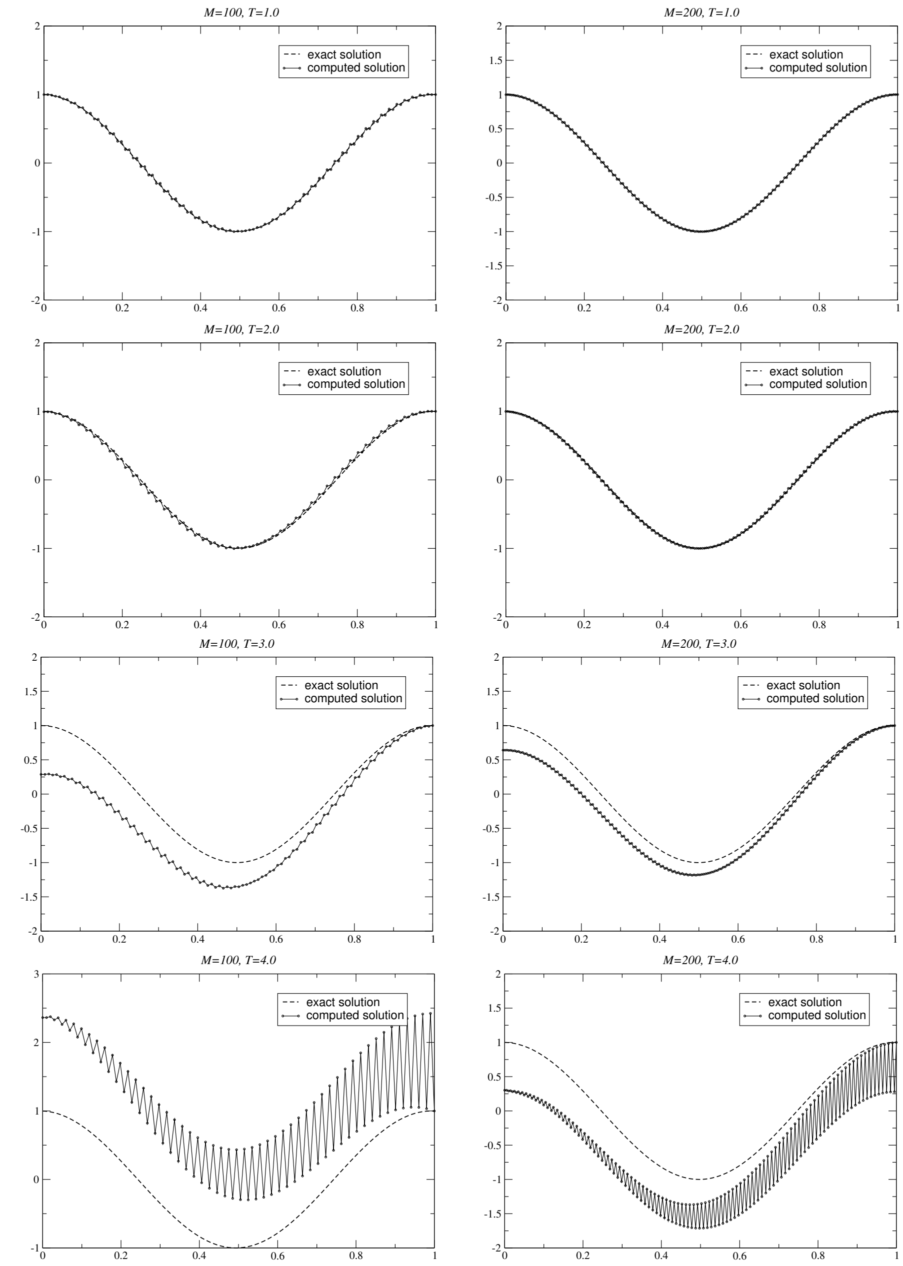

FIGURE 10.14: Solution of problem (10.183) with scheme (10.184), (10.185a).

one with the magnitude greater than one and the other with the magnitude less than one. When $|\lambda - 1| \longrightarrow 0$ we will clearly have $|q_1| \longrightarrow 1$ and $|q_2| \longrightarrow 1$. We, however, don't know ahead of time which of the two possible scenarios actually takes place:

$$\lim_{|\lambda|>1,\, \lambda\to 1} q_1(\lambda) = 1, \quad \lim_{|\lambda|>1,\, \lambda\to 1} q_2(\lambda) = -1 \tag{10.194a}$$

or

$$\lim_{|\lambda|>1,\, \lambda\to 1} q_1(\lambda) = -1, \quad \lim_{|\lambda|>1,\, \lambda\to 1} q_2(\lambda) = 1. \tag{10.194b}$$

To find this out, let us notice that the roots $q_1(\lambda)$ and $q_2(\lambda)$ are continuous (in fact, analytic) functions of λ. Consequently, if we take λ in the form $\lambda = 1+\eta$, where $|\eta| \ll 1$, and if we want to investigate the root q that is close to one, then we can say that $q(\lambda) = 1+\zeta$, where $|\zeta| \ll 1$. From equation (10.190a) we then obtain:

$$2\eta + \mathscr{O}(\eta^2) = 2r\zeta + \mathscr{O}(\zeta^2). \tag{10.195}$$

Consider a special case of real $\eta > 0$, then ζ must obviously be real as well. From the previous equality we find that $\zeta > 0$ (because $r > 0$), i.e., $|q| > 1$. As such, we see that if $|\lambda| > 1$ and $\lambda \longrightarrow 1$, then

$$\{q = q(\lambda) \longrightarrow 1\} \Longrightarrow \{|q| > 1\}.$$

Indeed, for real η and ζ, we have $|q| = 1+\zeta > 1$; for other η and ζ the same result follows by continuity. Consequently, it is the root q_2 that approaches $(1,0)$ when $\lambda \longrightarrow 1$, and the true scenario is given by (10.194b) rather than by (10.194a).

We therefore see that when a potentially "dangerous" unstable eigenvalue $|\lambda| > 1$ approaches the unit circle at $(1,0)$: $\lambda \longrightarrow 1$, it is the grid function $u_m = c_2 q_2^m$, $|q_2| > 1$, that will almost satisfy the boundary condition (10.185a), because $c_2(1-q_2) \longrightarrow 0$. This grid function, however, does not satisfy the requirement $u_m \longrightarrow 0$ as $m \longrightarrow \infty$, i.e., it does not belong to the class of functions admitted by problem (10.187). On the other hand, the function $u_m = c_1 q_1^m$, $|q_1| < 1$, that satisfies $u_m \longrightarrow 0$ as $m \longrightarrow \infty$, will be very far from satisfying the boundary condition (10.185a) because $q_1 \longrightarrow -1$.

Next, recall that we actually need to investigate what happens when $q_1 \longrightarrow 1$, i.e., when $c_1 q_1^m$ is almost an eigenfunction. This situation appears opposite to the one we have analyzed. Consequently, when $q_1 \longrightarrow 1$ we will not have such a $\lambda(q_1) \longrightarrow 1$ where $|\lambda(q_1)| > 1$. Qualitatively, this indicates that there is no instability associated with "almost an eigenfunction" $u_m = c_1 q_1^m$, $|q_1| < 1$, of problem (10.187). In the framework of the GKS theory, this assertion can be proven rigorously.

Let us now consider the second case: $\lambda \longrightarrow -1$ while $|\lambda| > 1$. We need to determine which of the two scenarios holds:

$$\lim_{|\lambda|>1,\, \lambda\to -1} q_1(\lambda) = 1, \quad \lim_{|\lambda|>1,\, \lambda\to -1} q_2(\lambda) = -1 \tag{10.196a}$$

or

$$\lim_{|\lambda|>1,\, \lambda\to -1} q_1(\lambda) = -1, \quad \lim_{|\lambda|>1,\, \lambda\to -1} q_2(\lambda) = 1. \tag{10.196b}$$

Similarly to the previous analysis, let $\lambda = -1+\eta$, where $|\eta| \ll 1$, then also $q(\lambda) = 1+\zeta$, where $|\zeta| \ll 1$ (recall, we are still interested in $q \longrightarrow 1$). Consider a particular case of real $\eta < 0$, then equation (10.195) yields $\zeta < 0$, i.e., $|q| < 1$. Consequently, if $|\lambda| > 1$ and $\lambda \longrightarrow -1$, then

$$\{q = q(\lambda) \longrightarrow 1\} \Longrightarrow \{|q| < 1\}.$$

In other words, this time it is the root q_1 that approaches $(1,0)$ as $\lambda \longrightarrow -1$, and the scenario that gets realized is (10.196a) rather than (10.196b). In contradistinction to the previous case, this presents a potential for instability. Indeed, the pair (λ, q_1), where $|q_1| < 1$ and $|\lambda| > 1$, would have implied the instability in the sense of Section 10.5.2 if $c_1 q_1^m$ were a genuine eigenfunction of problem (10.187) and λ if were the corresponding genuine eigenvalue. As we know, this is not the case. However, according to the first formula of (10.196a), the actual setup appears to be a limit of the admissible yet unstable situation. In other words, the combination of "almost an eigenfunction" $u_m = c_1 q_1^m$, $|q_1| < 1$, that satisfies $u_m \longrightarrow 0$ as $m \longrightarrow \infty$ with "almost an eigenvalue" $\lambda = \lambda(q_1)$, $|\lambda| > 1$, is unstable. While remaining unstable, this combination becomes more of a genuine eigenpair of problem (10.187) as $\lambda \longrightarrow -1$. Again, a rigorous proof of the instability is given in the framework of the GKS theory using the technique based on the Laplace transform.

Thus, we have seen that two scenarios are possible when λ approaches the unit circle from the outside. In one case, there may be an admissible root q of the characteristic equation that almost satisfies the boundary condition, see formula (10.196a), and this situation is prone to instability. Otherwise, see formula (10.194b), there is no admissible root q that would ultimately satisfy the boundary condition, and as such, no instability will be associated with this λ.

In the unstable case exemplified by formula (10.196a), the corresponding limit value of λ is called *the generalized eigenvalue,* see [GKO95, Chapter 13]. In particular, $\lambda = -1$ is a generalized eigenvalue of problem (10.187). We re-emphasize that it is not a genuine eigenvalue of problem (10.187), because when $\lambda = -1$ then $q_1 = 1$ and the eigenfunction $u_m = cq_1^m$ does not belong to the admissible class: $u_m \longrightarrow 0$ as $m \longrightarrow \infty$. In fact, it is easy to see that $\|u\|_2 = \infty$. However, it is precisely this generalized eigenvalue that causes the instability even when the entire spectrum of the family of operators $\{\boldsymbol{R}_h\}$ belongs to the unit disk and $r < 1$.

Accordingly, *the Kreiss necessary and sufficient condition of stability* requires that the spectrum of the family of operators be confined to the unit disk as before, and additionally, that the scheme should have no generalized eigenvalues $|\lambda| = 1$. In the case of systems, the discrete Cauchy problem must also be stable in the sense of Theorem 10.4 (which, for the leap-frog scheme, means $r < 1$). Scheme (10.184), (10.185a) violates the Kreiss condition as it has a generalized eigenvalue $\lambda = -1$. Hence, it is unstable, see Figure 10.14.

Since, however, this instability is only due to a generalized eigenvalue with $|\lambda| = 1$, it is relatively mild, as expected. On the other hand, if we were to replace the marginally unstable boundary condition (10.185a) with a truly unstable one in the sense of Section 10.5.2, then the effect on the stability of the scheme would have been much more drastic. Instead of (10.185a), consider, for example:

$$u_0^{p+1} = 1.05 \cdot u_1^{p+1}. \tag{10.197}$$

This boundary condition generates an eigenfunction $u_m = c_1 q_1^m$ of problem (10.187) with $q_1 = \frac{1}{1.05} < 1$. The corresponding eigenvalues are given by:

$$\lambda(q_1) = \frac{r}{2}\left(q_1 - \frac{1}{q_1}\right) \pm \sqrt{1 + \frac{r^2}{4}\left(q_1 - \frac{1}{q_1}\right)^2},$$

and for one of these eigenvalues we obviously have $|\lambda| > 1$. Therefore, the scheme is unstable according to Theorem 10.8, see also Figure 10.15.

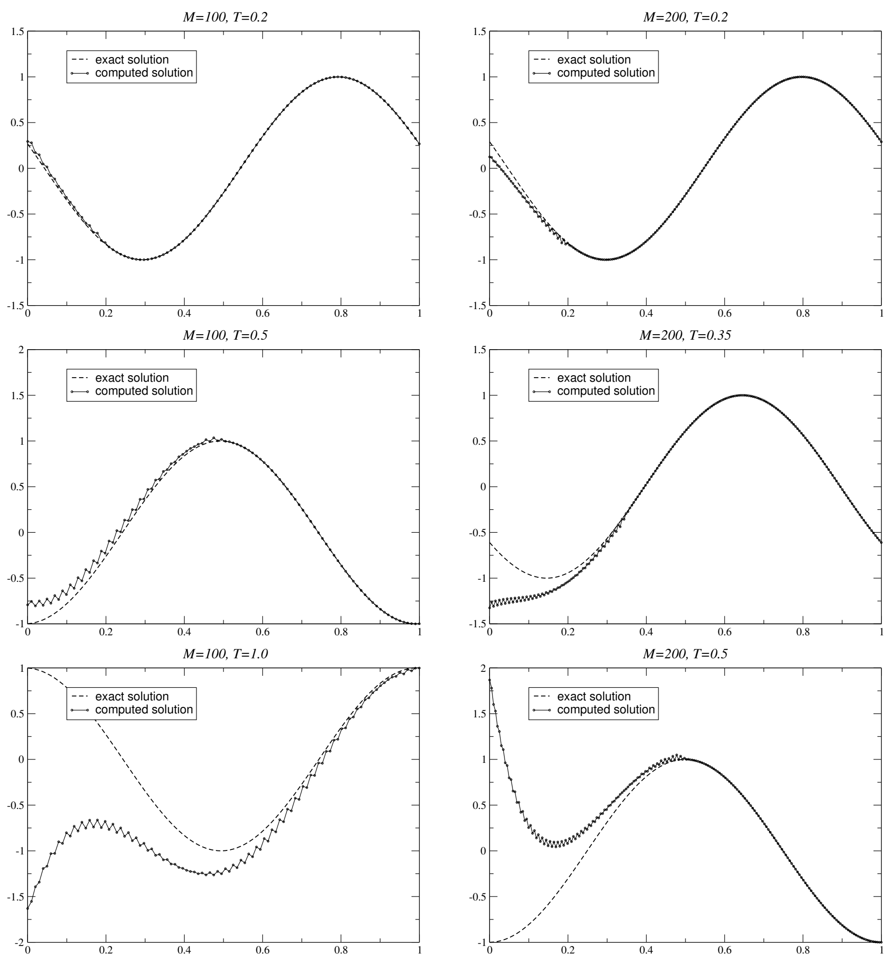

FIGURE 10.15: Solution of problem (10.183) with scheme (10.184), (10.197).

In Figure 10.15, we are showing the results of the numerical solution of problem (10.183) using the unstable scheme (10.184), (10.197). Comparing the plots in Figure 10.15 with those in Figure 10.14, we see that in the case of boundary condition (10.197) the instability develops much more rapidly in time. Moreover, comparing the left column in Figure 10.15 that corresponds to the grid with $M = 100$ cells with the right column in the same figure that corresponds to $M = 200$, we see that the instability develops more rapidly on a finer grid, which is characteristic of an exponential instability.

Let now now analyze the second outflow boundary condition (10.185b):

$$u_0^{p+1} = u_0^p + r(u_1^p - u_0^p).$$

Unlike the extrapolation-type boundary condition (10.185a), which to some extent is arbitrary, boundary condition (10.185b) merely coincides with the first order upwind approximation of the differential equation itself that we have encountered previously on multiple occasions. To study stability, we again need to investigate three model problems: (10.186), (10.187), and (10.188). Obviously, only problem (10.187) changes due to the new boundary condition, where the other two stay the same. Moreover, as the Cauchy problem (10.186) is not stable for $r = 1$, it is sufficient to analyze the boundary conditions only for $r < 1$.

To find λ and q for problem (10.187), we need to solve the characteristic equation (10.190a) along with a similar equation that comes from the boundary condition (10.185b):

$$\lambda = 1 - r + rq. \tag{10.198}$$

Substituting λ from equation (10.198) into equation (10.190a) and subsequently solving for q, we find that there is only one solution: $q = 1$. For the corresponding λ, we then have from equation (10.198): $\lambda = 1$. Consequently, for $r < 1$ problem (10.187) has no proper eigenfunctions/eigenvalues, which means that we again have $\overrightarrow{\Lambda} = \emptyset$. As far as the generalized eigenvalues, we only need to check one value of λ: $\lambda = 1$ (because $\lambda = -1$ does not satisfy equation (10.198) for $q = 1$). Let $\lambda = 1 + \eta$ and $q = 1 + \zeta$, where $|\eta| \ll 1$ and $|\zeta| \ll 1$. We then arrive at the same equation (10.195) that we obtained in the context of the previous analysis and conclude that $\lambda = 1$ does not violate the Kreiss condition, because $|\lambda| > 1$ implies $|q| > 1$. As such, the scheme (10.184), (10.185b) is stable when $r < 1$.

Exercises

1. For the scalar Lax-Wendroff scheme [cf. formula (10.83)]:

$$\frac{u_m^{p+1} - u_m^p}{\tau} - \frac{u_{m+1}^p - u_{m-1}^p}{2h} - \frac{\tau}{2}\frac{u_{m+1}^p - 2u_m^p + u_{m-1}^p}{h^2} = 0,$$
$$p = 0, 1, \dots, [T/\tau] - 1, \quad m = 1, 2, \dots, M-1, \quad Mh = 1,$$
$$u_m^0 = \psi(x_m), \quad m = 0, 1, 2, \dots, M,$$
$$\frac{u_0^{p+1} - u_0^p}{\tau} - \frac{u_1^p - u_0^p}{h} = 0, \quad u_M^{p+1} = 0, \quad p = 0, 1, \dots, [T/\tau] - 1,$$

that approximates the initial boundary value problem:

$$\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} = 0, \quad 0 \le x \le 1, \quad 0 < t \le T,$$
$$u(x,0) = \psi(x), \quad u(1,t) = 0,$$

on the uniform rectangular grid: $x_m = mh$, $m = 0,1,\dots,M$, $Mh = 1$, $t_p = p\tau$, $p = 0,1,\dots,[T/\tau]$, find out when the Babenko-Gelfand stability criterion holds.

Answer. $r = \tau/h \le 1$.

2.* Prove Theorem 10.6.

a) Prove the sufficiency part.
b) Prove the necessity part.

3.* Approximate the acoustics Cauchy problem:

$$\frac{\partial \boldsymbol{u}}{\partial t} - \boldsymbol{A}\frac{\partial \boldsymbol{u}}{\partial x} = \boldsymbol{\varphi}(x,t), \quad -\infty \le x \le \infty, \quad 0 < t \le T,$$
$$\boldsymbol{u}(x,0) = \boldsymbol{\psi}(x), \quad -\infty \le x \le \infty,$$
$$\boldsymbol{u}(x,t) = \begin{bmatrix} v(x,t) \\ w(x,t) \end{bmatrix}, \quad \boldsymbol{\varphi}(x) = \begin{bmatrix} \varphi^{(1)}(x) \\ \varphi^{(2)}(x) \end{bmatrix}, \quad \boldsymbol{\psi}(x) = \begin{bmatrix} \psi^{(1)}(x) \\ \psi^{(2)}(x) \end{bmatrix}, \quad \boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

with the Lax-Wendroff scheme:

$$\frac{\boldsymbol{u}_m^{p+1} - \boldsymbol{u}_m^p}{\tau} - \boldsymbol{A}\frac{u_{m+1}^p - u_{m-1}^p}{2h} - \frac{\tau}{2}\boldsymbol{A}^2\frac{\boldsymbol{u}_{m+1}^p - 2\boldsymbol{u}_m^p + \boldsymbol{u}_{m-1}^p}{h^2} = \boldsymbol{\varphi}_m^p,$$
$$p = 0,1,\dots,[T/\tau]-1, \quad m = 0,\pm 1,\pm 2,\dots,$$
$$\boldsymbol{u}_m^0 = \boldsymbol{\psi}(x_m), \quad m = 0,\pm 1,\pm 2,\dots.$$

Define $\boldsymbol{u}^p = \{\boldsymbol{u}_m^p\}$ and $\boldsymbol{\varphi}^p = \{\boldsymbol{\varphi}_m^p\}$, and introduce the norms as follows:

$$\|\boldsymbol{u}^{(h)}\|_{U_h} = \max_p \|\boldsymbol{u}^p\|, \quad \|\boldsymbol{f}^{(h)}\|_{F_h} = \max\left[\|\boldsymbol{\psi}\|, \max_p \|\boldsymbol{\varphi}^p\|\right],$$

where

$$\|\boldsymbol{u}^p\|^2 = \sum_m \left(|v_m^p|^2 + |w_m^p|^2\right), \quad \|\boldsymbol{\psi}\|^2 = \sum_m \left(|\psi^{(1)}(x_m)|^2 + |\psi^{(2)}(x_m)|^2\right),$$
$$\|\boldsymbol{\varphi}^p\|^2 = \sum_m \left(|\varphi^{(1)}(x_m,t_p)|^2 + |\varphi^{(2)}(x_m,t_p)|^2\right).$$

a) Show that when reducing the Lax-Wendroff scheme to the canonical form (10.141), inequalities (10.143) and (10.144) hold.
b) Prove that when $r = \frac{\tau}{h} \le 1$ the scheme is l_2 stable, and when $r > 1$ it is unstable.

Hint. To prove estimate (10.145) for the norms $\|\boldsymbol{R}_h^p\|$, first introduce the new unknown variables (called the Riemann invariants):

$$I_m^{(1)} = v_m + w_m \text{ and } I_m^{(2)} = v_m - w_m,$$

and transform the discrete system accordingly, and then employ the spectral criterion of Section 10.3.

4. Let the norm in the space U'_h be defined in the sense of l_2: $\|u\|_2 = \left[h \sum\limits_{m=-\infty}^{\infty} |u_m|^2\right]^{1/2}$. Prove that in this case all complex numbers $\lambda(\alpha) = 1 - r + re^{i\alpha}$, $0 \le \alpha < 2\pi$ [see formula (10.148)], belong to the spectrum of the transition operator $\boldsymbol{R}_h$ that corresponds to the difference Cauchy problem (10.147), where the spectrum is defined according to Definition 10.7.

   Hint. Construct the solution $u = \{u_m\}$, $m = 0, \pm 1, \pm 2, \dots$, to the inequality that appears in Definition 10.7 in the form: $u_m = \begin{cases} q_1^m, & m \ge 0, \\ q_2^{-m}, & m < 0, \end{cases}$, where $q_1 = (1-\delta)e^{i\alpha}$, $q_2 = (1-\delta)e^{-i\alpha}$, and $\delta > 0$ is a small quantity.

5. Prove sufficiency in Theorem 10.7.

   Hint. Use expansion with respect to an orthonormal basis in U' composed of the eigenvectors of $\boldsymbol{R}_h$.

6. Compute the spectrum of the family of operators $\{\boldsymbol{R}_h\}$, $v = \boldsymbol{R}_h u$, given by the formulae:

$$\begin{aligned} v_m &= (1-r)u_m + ru_{m+1}, \quad m = 0, 1, \dots, M-1, \\ v_M &= 0. \end{aligned}$$

   Assume that the norm is the maximum norm.

7. Prove that the spectrum of the family of operators $\{\boldsymbol{R}_h\}$, $v = \boldsymbol{R}_h u$, defined as:

$$\begin{aligned} v_m &= (1-r+\gamma h)u_m + ru_{m+1}, \quad m = 0, 1, \dots, M-1, \\ v_M &= u_M, \end{aligned}$$

   does not depend on the value of γ and coincides with the spectrum computed in Section 10.5.2 for the case $\gamma = 0$. Assume that the norm is the maximum norm.

   Hint. Notice that this operator is obtained by adding $\gamma h \boldsymbol{I}'$ to the operator $\boldsymbol{R}_h$ defined by formulae (10.142a) & (10.142b), and then use Definition 10.6 directly. Here $\boldsymbol{I}'$ is a modification of the identity operator that leaves all components of the vector u intact except the last component u_M that is set to zero.

8. Compute the spectrum of the family of operators $\{\boldsymbol{R}_h\}$, $v = \boldsymbol{R}_h u$, given by the formulae:

$$\begin{aligned} v_m &= (1-r)u_m + r(u_{m-1} + u_{m+1})/2, \quad m = 1, 2, \dots, M-1, \\ v_M &= 0, \quad av_0 + bv_1 = 0, \end{aligned}$$

   where $a \in \mathbb{R}$ and $b \in \mathbb{R}$ are known and fixed. Consider the cases $|a| > |b|$ and $|a| < |b|$.

9.* Prove that the spectrum of the family of operators $\{\boldsymbol{R}_h\}$, $v = \boldsymbol{R}_h u$, defined by formulae (10.142a) & (10.142b) and analyzed in Section 10.5.2:

$$\begin{aligned} v_m &= (1-r)u_m + ru_{m+1}, \quad m = 0, 1, \dots, M-1, \\ v_M &= u_M, \end{aligned}$$

   will not change if the C norm: $\|u\| = \max_m |u_m|$ is replaced by the l_2 norm: $\|u\| = \left[h \sum_m u_m^2\right]^{1/2}$.

10. For the first order ordinary difference equation:

$$av_m + bv_{m+1} = f_m, \quad m = 0, \pm 1, \pm 2, \ldots,$$

the fundamental solution G_m is defined as *a bounded solution* of the equation:

$$aG_m + bG_{m+1} = \delta_m \equiv \begin{cases} 1, & m = 0, \\ 0, & m \neq 0. \end{cases}$$

a) Prove that if $|a/b| < 1$, then $G_m = \begin{cases} 0, & m \leq 0, \\ -\frac{1}{a}\left(-\frac{a}{b}\right)^m, & m \geq 1. \end{cases}$

b) Prove that if $|a/b| > 1$, then $G_m = \begin{cases} \frac{1}{a}\left(-\frac{a}{b}\right)^m, & m \leq 0, \\ 0, & m \geq 1. \end{cases}$

c) Prove that $v_m = \sum\limits_{k=-\infty}^{\infty} G_{m-k} f_k$.

11. Obtain energy estimates for the implicit first order upwind schemes that approximate problems (10.170), (10.172)*, and (10.174)*.

12.* Approximate problem (10.170) with the Crank-Nicolson scheme supplemented by one-sided differences at the left boundary $x = 0$:

$$\begin{gathered} \frac{u_m^{p+1} - u_m^p}{\tau} - \frac{1}{2}\left[\frac{u_{m+1}^{p+1} - u_{m-1}^{p+1}}{2h} + \frac{u_{m+1}^p - u_{m-1}^p}{2h}\right] = 0, \\ m = 1, 2, \ldots, M-1, \quad p = 0, 1, \ldots, [T/\tau] - 1, \\ \frac{u_0^{p+1} - u_0^p}{\tau} - \frac{1}{2}\left[\frac{u_1^{p+1} - u_0^{p+1}}{h} + \frac{u_1^p - u_0^p}{h}\right] = 0, \quad u_M^p = 0, \\ p = 0, 1, \ldots, [T/\tau] - 1, \\ u_m^0 = \psi_m, \quad m = 0, 1, 2, \ldots, M. \end{gathered} \tag{10.199}$$

a) Use an alternative definition of the l_2 norm: $\|u\|_2^2 = \frac{h}{2}(u_0^2 + u_M^2) + h\sum\limits_{m=1}^{M-1} u_m^2$ and develop an energy estimate for scheme (10.199).

Hint. Multiply the equation by $u_m^{p+1} + u_m^p$ and sum over the entire range of m.

b) Construct the schemes similar to (10.199) for the variable-coefficient problems (10.172) and (10.174) and obtain energy estimates.

13. Using the Kreiss condition, show that the leap-frog scheme (10.184) with the boundary condition:

$$u_0^{p+1} = u_1^p \tag{10.200a}$$

is stable provided that $r < 1$, whereas with the boundary condition:

$$u_0^{p+1} = u_0^{p-1} + 2r(u_1^p - u_0^p) \tag{10.200b}$$

it is unstable.