the spatial term:

$$\frac{d}{dt}\int_0^1 \frac{u^2(x,t)}{2}dx - \int_0^1 a(x,t)\frac{\partial}{\partial x}\frac{u^2(x,t)}{2}dx$$
$$= \frac{d}{dt}\frac{\|u(\cdot,t)\|_2^2}{2} - a(1,t)\frac{u^2(1,t)}{2} + a(0,t)\frac{u^2(0,t)}{2} + \int_0^1 a'_x(x,t)\frac{u^2(x,t)}{2}dx = 0.$$

Using the boundary condition $u(1,t)=0$, we find:

$$\frac{d}{dt}\frac{\|u(\cdot,t)\|_2^2}{2} = -a(0,t)\frac{u^2(0,t)}{2} - \int_0^1 a'_x(x,t)\frac{u^2(x,t)}{2}dx.$$

The first term on the right-hand side of the previous equality is always non-positive. As far as the second term, let us denote $A = \sup_{(x,t)}[-a'_x(x,t)]$. Then we have:

$$\frac{d}{dt}\|u(\cdot,t)\|_2^2 \le A\|u(\cdot,t)\|_2^2,$$

which immediately yields:

$$\|u(\cdot,t)\|_2 \le e^{At/2}\|\psi\|_2, \quad t \ge 0.$$

If $A<0$, the previous inequality implies that the L_2 norm of the solution decays as $t \longrightarrow +\infty$. If $A=0$, then the norm of the solution stays bounded by the norm of the initial data. To obtain an overall uniform estimate of $\|u(\cdot,t)\|_2$ for $A \le 0$ and all $t \ge 0$, we need to select the maximum value of the constant: $\max_t e^{At/2} = 1$, and then the desired inequality will coincide with (10.171). For $A>0$, a uniform estimate can only be obtained for a given fixed interval $0 \le t \le T$, so that altogether we can write:

$$\|u(\cdot,t)\|_2 \le \begin{cases} \|\psi\|_2, & \text{if } A \le 0, \ \ t \ge 0, \\ e^{AT/2}\|\psi\|_2, & \text{if } A > 0, \ \ 0 \le t \le T. \end{cases} \tag{10.173}$$

Similarly to inequality (10.171), the energy estimate (10.173) also provides a bound for the L_2 norm of the solution in terms of the L_2 norm of the initial data. However, when $A>0$ the constant in front of $\|\psi\|_2$ is no longer equal to one. Instead, $e^{AT/2}$ grows exponentially as the maximum time T elapses, and therefore estimate (10.173) for $A>0$ may only be considered on a finite interval $0 \le t \le T$ rather than for $t \ge 0$.

In problems (10.170) and (10.172) the boundary condition at $x=1$ was homogeneous. Let us now introduce yet another generalization and analyze the problem:

$$\begin{gathered} \frac{\partial u}{\partial t} - a(x,t)\frac{\partial u}{\partial x} = 0, \quad 0 \le x \le 1, \quad 0 < t \le T, \\ u(x,0) = \psi(x), \quad u(1,t) = \chi(t), \end{gathered} \tag{10.174}$$

that differs from (10.172) by its inhomogeneous boundary condition: $u(1,t) = \chi(t)$. Otherwise everything is the same; in particular, we still assume that $\forall x \in [0,1]$ and $\forall t \geq 0:\ a(x,t) \geq a_0 > 0$ and denote $A = \sup_{(x,t)}[-a'_x(x,t)]$. Multiplying the differential equation of (10.174) by $u(x,t)$ and integrating by parts, we obtain:

$$\frac{d}{dt}\frac{\|u(\cdot,t)\|_2^2}{2} = -a(0,t)\frac{u^2(0,t)}{2} + a(1,t)\frac{\chi^2(t)}{2} - \int_0^1 a'_x(x,t)\frac{u^2(x,t)}{2}dx.$$

Consequently,

$$\frac{d}{dt}\|u(\cdot,t)\|_2^2 \leq A\|u(\cdot,t)\|_2^2 + a(1,t)\chi^2(t).$$

Multiplying the previous inequality by e^{-At}, we have:

$$\frac{d}{dt}\left[e^{-At}\|u(\cdot,t)\|_2^2\right] \leq e^{-At}a(1,t)\chi^2(t),$$

which, after integrating over the time interval $0 \leq \theta \leq t$, yields:

$$\|u(\cdot,t)\|_2^2 \leq e^{At}\|\psi\|_2^2 + e^{At}\int_0^t e^{-A\theta}a(1,\theta)\chi^2(\theta)d\theta.$$

As in the case of a homogeneous boundary condition, we would like to obtain a uniform energy estimate for a given interval of time. This can be done if we again distinguish between $A \leq 0$ and $A > 0$. When $A \leq 0$ we can consider all $t \geq 0$ and when $A > 0$ we can only have the estimate on some fixed interval $0 \leq t \leq T$:

$$\|u(\cdot,t)\|_2^2 \leq \begin{cases} \|\psi\|_2^2 + \int_0^\infty a(1,\theta)\chi^2(\theta)d\theta, & A \leq 0,\ \ t \geq 0, \\ e^{AT}\left[\|\psi\|_2^2 + \int_0^T a(1,\theta)\chi^2(\theta)d\theta\right], & A > 0,\ \ 0 \leq t \leq T. \end{cases} \tag{10.175}$$

When deriving inequalities (10.175), we obviously need to assume that the integrals on the right-hand side of (10.175) are bounded. These integrals can be interpreted as weighted L_2 norms of the boundary data $\chi(t)$. Clearly, energy estimate (10.175) includes the previous estimate (10.173) as a particular case for $\chi(t) \equiv 0$.

All three estimates (10.171), (10.173), and (10.175) indicate that the corresponding initial boundary value problem is *well-posed* in the sense of L_2. Qualitatively, well-posedness means that the solution to a given problem is only weakly sensitive to perturbations of the input data (such as initial and/or boundary data). In the case of linear evolution problems, well-posedness can, for example, be quantified by means of the energy estimates. These estimates provide a bound for the norm of the solution in terms of the norms of the input data. In the finite-difference context, similar estimates would have implied stability in the sense of l_2, provided that the corresponding constants on the right-hand side of each inequality can be chosen independent of the grid. We will now proceed to demonstrate how energy estimates can be obtained for finite-difference schemes.