where v_m are components of the bounded solution $v = \{v_m\}$, $m = 0, \pm 1, \pm 2, \ldots$, to the following equation:

$$(1 - r - \lambda_0)v_m + rv_{m+1} = \hat{f}_m \stackrel{\text{def}}{=} \begin{cases} 0, & \text{if } m < 0, \\ f_m, & \text{if } m = 0, 1, \ldots, M-1, \\ 0, & \text{if } m \geq M. \end{cases} \tag{10.166}$$

Then because of the linearity, the grid function $w = \{w_m\}$ introduced by formula (10.165) solves the equation:

$$\begin{gathered} (1 - r - \lambda_0)w_m + rw_{m+1} = 0, \qquad m = 0, 1, \ldots, M-1, \\ (1 - \lambda_0)w_M = f_M - (1 - \lambda_0)v_M. \end{gathered} \tag{10.167}$$

Let us now recast estimate (10.164) as $|u_m| \leq A^{-1} \max\limits_m |f_m|$. According to (10.165), to prove this estimate it is sufficient to establish individual inequalities:

$$|v_m| \leq A_1 \max_m |f_m|, \tag{10.168a}$$

$$|w_m| \leq A_2 \max_m |f_m|, \tag{10.168b}$$

where A_1 and A_2 are constants. We begin with inequality (10.168a). Notice that equation (10.166) is a first order constant-coefficient ordinary difference equation:

$$av_m + bv_{m+1} = \hat{f}_m, \quad m = 0, \pm 1, \pm 2, \ldots,$$

where $a = 1 - r - \lambda_0$, $b = r$. Its bounded fundamental solution is given by

$$G_m = \begin{cases} \dfrac{1}{a}\left(-\dfrac{a}{b}\right)^m, & m \leq 0, \\ 0, & m \geq 1, \end{cases}$$

because $\lambda_0 \notin \{\overleftrightarrow{\Lambda} \cup \overleftarrow{\Lambda} \cup \overrightarrow{\Lambda}\}$, i.e., $|\lambda_0 - (1 - r)| > r$, and consequently $|a/b| > 1$. Representing the solution v_m in the form of a convolution: $v_m = \sum\limits_{k=-\infty}^{\infty} G_{m-k}\hat{f}_k$ and summing up the geometric sequence we arrive at the estimate:

$$|v_m| \leq \frac{\max\limits_m |\hat{f}_m|}{|a| - |b|} \leq \frac{\max\limits_m |f_m|}{|a| - |b|}.$$

Introducing the distance δ_0 between the point λ_0 and the set $\{\overleftrightarrow{\Lambda} \cup \overleftarrow{\Lambda} \cup \overrightarrow{\Lambda}\}$, we can obviously claim that $|a| - |b| > \delta_0/2$, which makes the previous estimate equivalent to (10.168a). Estimate (10.168b) can be obtained by representing the solution of equation (10.167) in the form:

$$w_m = \frac{f_M - (1 - \lambda_0)v_M}{1 - \lambda_0} q_0^{m-M}, \tag{10.169}$$

where q_0 is determined by the relation $(1-r-\lambda_0)+rq_0=0$. Our assumption is that $\lambda_0 \notin \{\overleftrightarrow{\Lambda} \cup \overleftarrow{\Lambda} \cup \overrightarrow{\Lambda}\}$, i.e., that λ_0 lies outside of the disk of radius r on the complex plane centered at $(1-r,0)$. In this case $|q_0|>1$. Moreover, we can say that $|1-\lambda_0|=\delta_1>0$, because if $\lambda_0=1$, then λ_0 would have belonged to the set $\{\overleftrightarrow{\Lambda} \cup \overleftarrow{\Lambda} \cup \overrightarrow{\Lambda}\}$. As such, using formula (10.169) and taking into account estimate (10.168a) that we have already proved, we obtain the desired estimate (10.168b):

$$\begin{aligned}|w_m| &= \left|\frac{f_M-(1-\lambda_0)v_M}{1-\lambda_0}\right| \cdot |q_0^{m-M}| \leq \frac{|f_M|}{|1-\lambda_0|}+|v_M| \\ &\leq \frac{\max\limits_m |f_m|}{\delta_1}+A_1\max_m |f_m| = A_2 \max_m |f_m|.\end{aligned}$$

We have thus proven that the spectrum of the family of operators $\{\boldsymbol{R}_h\}$ defined by formulae (10.142) coincides with the set $\{\overleftrightarrow{\Lambda} \cup \overleftarrow{\Lambda} \cup \overrightarrow{\Lambda}\}$ on the complex plane.

The foregoing algorithm for computing the spectrum of the family of operators $\{\boldsymbol{R}_h\}$ is, in fact, quite general. We have illustrated it using a particular example of the operators defined by formulae (10.142). However, not only for this specific example but also for other scalar one-step finite-difference schemes with constant coefficients, the spectrum of the family of operators $\{\boldsymbol{R}_h\}$ can be obtained by performing the same Babenko-Gelfand analysis of Section 10.5.1. *The key idea is to take into account other candidate modes that may be prone to developing the instability, besides the eigenmodes* $\{e^{i\alpha m}\}$ *of the pure Cauchy problem that are accounted for by the von Neumann analysis.*

For systems of finite-difference equations (as well as for scalar multi-step equations), the technical side of the procedure may become more involved. In this case, the computation of the spectrum of a family of operators can be reduced to studying uniform bounds for the solutions of certain ordinary difference equations with matrix coefficients. A necessary and sufficient condition has been obtained in [Rya64] for the existence of such uniform bounds. This condition is given in terms of the roots of the corresponding characteristic equation and also involves the analysis of some determinants originating from the matrix coefficients of the system. For further detail, we refer the reader to [GR87, § 4 & § 45] and [RM67, § 6.6 & § 6.7], as well as to the original journal publication by Ryaben'kii [Rya64].

10.5.3 The Energy Method

For some evolution finite-difference problems, one can obtain the l_2 estimates of the solution directly, i.e., without employing any special stability criteria, such as spectral. The corresponding technique is known as the method of energy estimates. It is useful for deriving *sufficient conditions of stability,* in particular, because it can often be applied to problems with variable coefficients on finite intervals. We illustrate the energy method with several examples.

In the beginning, let us analyze the continuous case. Consider an initial boundary