10.3.4 Stability in C

Let us emphasize that the type of stability we have analyzed in Sections 10.3.1–10.3.3 was stability in the sense of the maximum norm (10.71). Alternatively, it is referred to as stability in (the space) C. This space contains all bounded numerical sequences. The von Neumann spectral condition (10.78) is *necessary* for the scheme to be stable in C. As far as the sufficient conditions, in some simple cases stability in C can be proved directly, for example, using maximum principle, as done in Section 10.1.3 for the first order explicit upwind scheme and in Section 10.6.1 for an explicit scheme for the heat equation. Otherwise, sufficient conditions for stability in C turn out to be delicate and may require rather sophisticated arguments. The analysis of a general case even for one scalar constant coefficient difference equation goes beyond the scope of the current book, and we refer the reader to the original work by Fedoryuk [Fed67] (see also his monograph [Fed77, Chapter V, § 4]). In addition, in [RM67, Chapter 5] the reader can find an account of the work by Strang and by Thomee on the subject.

10.3.5 Sufficiency of the Spectral Stability Condition in l_2

However, *a sufficient condition for stability* may sometimes be easier to find if we were to use a different norm instead of the maximum norm (10.71). Let, for example,

$$\begin{gathered}\|u^p\|^2 = \sum_{m=-\infty}^{\infty} |u_m^p|^2, \quad \|\varphi^p\|^2 = \sum_{m=-\infty}^{\infty} |\varphi_m^p|^2, \quad \|\psi\|^2 = \sum_{m=-\infty}^{\infty} |\psi_m|^2, \\ \|u^{(h)}\|_{U_h} = \max_p \|u^p\|, \quad \|f^{(h)}\|_{F_h} = \left\| \begin{matrix}\varphi^p \\ \psi\end{matrix} \right\|_{F_h} = \max\{\|\psi\|, \max_p \|\varphi^p\|\}.\end{gathered} \tag{10.94}$$

Relations (10.94) define Euclidean (i.e., l_2) norms for u^p, φ^p, and ψ. Accordingly, stability in the sense of the norms given by (10.94) is referred to as stability in (the space) l_2. We recall that the space l_2 is a Hilbert space of all numerical sequences, for which the sum of squares of absolute values of all their terms is bounded.

Consider a general constant coefficient finite-difference Cauchy problem:

$$\begin{gathered}\sum_{j=-j_{\text{left}}}^{j_{\text{right}}} b_j u_{m+j}^{p+1} - \sum_{j=-j_{\text{left}}}^{j_{\text{right}}} a_j u_{m+j}^p = \varphi_m^p, \\ u_m^0 = \psi_m, \quad m = 0, \pm 1, \pm 2, \dots, \quad p = 0, 1, \dots, [T/\tau] - 1,\end{gathered} \tag{10.95}$$

under the assumption that

$$\sum_{j=-j_{\text{left}}}^{j_{\text{right}}} b_j e^{i\alpha j} \neq 0, \quad 0 \le \alpha < 2\pi.$$

Note that all spatially one-dimensional schemes from Section 10.3.3, except those in Examples 5 and 9, fit into the category (10.95) with $j_{\text{left}} = j_{\text{right}} = 1$.

THEOREM 10.3

For the scheme (10.95) to be stable in l_2 with respect to the initial data, i.e., for the following inequality to hold:

$$\|u^p\| \leq c\|\psi\|, \quad p = 0, 1, \dots, [T/\tau], \tag{10.96}$$

where the constant c does not depend on h [or on $\tau = \tau(h)$], it is necessary and sufficient that the von Neumann condition (10.78) be satisfied, i.e., that the spectrum of the scheme $\lambda = \lambda(\alpha)$ belong to the disk:

$$|\lambda(\alpha)| \leq 1 + c_1\tau, \tag{10.97}$$

where c_1 is another constant that does not depend either on α or on τ.

PROOF We will first prove the sufficiency. By hypotheses of the theorem, the number series $\sum\limits_{m=-\infty}^{\infty} |\psi_m|^2$ converges. Then, the function series of the independent variable α:

$$\sum_{m=-\infty}^{\infty} \psi_m e^{-i\alpha m}$$

also converges in the space $L_2[0, 2\pi]$, and its sum that we denote $\Psi(\alpha)$, $0 \leq \alpha \leq 2\pi$, is a function that has ψ_m as the Fourier coefficients:

$$\psi_m = \frac{1}{2\pi} \int_0^{2\pi} \Psi(\alpha) e^{i\alpha m} d\alpha, \quad m = 0, \pm 1, \pm 2, \dots, \tag{10.98}$$

(a realization of the Riesz-Fischer theorem, see, e.g., [KF75, Section 16]). Moreover, the following relation holds:

$$\|\psi\|^2 = \sum_{m=-\infty}^{\infty} |\psi_m|^2 = \frac{1}{2\pi} \int_0^{2\pi} |\Psi(\alpha)|^2 d\alpha = \frac{1}{2\pi} \|\Psi\|_2^2$$

known as the Parseval equality.

Consider a homogeneous counterpart to the difference equation (10.95):

$$\sum_{j=-j_{\text{left}}}^{j_{\text{right}}} b_j u_{m+j}^{p+1} - \sum_{j=-j_{\text{left}}}^{j_{\text{right}}} a_j u_{m+j}^p = 0,$$
$$m = 0, \pm 1, \pm 2, \dots, \quad p = 0, 1, \dots, [T/\tau] - 1,$$

For any $\alpha \in [0, 2\pi)$ this equation obviously has a solution of the type:

$$u_m^p(\alpha) = \lambda^p(\alpha) e^{i\alpha m} \tag{10.99}$$

for some particular $\lambda = \lambda(\alpha)$ that can be determined by substitution:

$$\lambda(\alpha) = \left(\sum_{j=-j_{\text{left}}}^{j_{\text{right}}} a_j e^{i\alpha j}\right)\left(\sum_{j=-j_{\text{left}}}^{j_{\text{right}}} b_j e^{i\alpha j}\right)^{-1}.$$

Introduce the grid function:

$$u_m^p = \frac{1}{2\pi}\int_0^{2\pi} \Psi(\alpha)\lambda^p(\alpha)e^{i\alpha m}d\alpha, \quad m = 0, \pm 1, \pm 2, \dots \tag{10.100}$$

If the von Neumann spectral condition (10.97) holds, we have:

$$|\lambda(\alpha)|^p \le |1 + c_1\tau|^{T/\tau} \le e^{c_1 T}. \tag{10.101}$$

Then, the integral on the right-hand side of (10.100) indeed converges, because $\Psi(\alpha) \in L_2$ along with (10.101) imply $\Psi(\alpha)\lambda^p(\alpha) \in L_2$, and $\int_0^{2\pi} |\Psi(\alpha)\lambda^p(\alpha)|^2 d\alpha < \infty \Longrightarrow \int_0^{2\pi} |\Psi(\alpha)\lambda^p(\alpha)| d\alpha < \infty$.

The function u_m^p of (10.100) solves the Cauchy problem (10.95) for $\varphi_m^p = 0$ because it is a linear combination of solutions $u_m^p(\alpha)$ of (10.99) and coincides with ψ_m for $p = 0$, see (10.98). Consequently, using the Parseval equality and inequality (10.101), we can obtain:

$$\|u^p\|^2 = \frac{1}{2\pi}\int_0^{2\pi} |\lambda^p(\alpha)\Psi(\alpha)|^2 d\alpha \le e^{2c_1 T}\frac{1}{2\pi}\int_0^{2\pi} |\Psi(\alpha)|^2 d\alpha = e^{2c_1 T}\|\psi\|^2,$$

which clearly implies stability with respect to the initial data: $\|u^p\| \le c\|\psi\|$.

To prove the necessity, we will need to show that if (10.97) holds for no fixed c_1, then the scheme is unstable. We should emphasize that to demonstrate the instability for the chosen norm (10.94) we may not exploit the unboundedness of the solution $u_m^p(\alpha) = \lambda^p(\alpha)e^{i\alpha m}$ that takes place in this case, because the grid function $\{e^{i\alpha m}\}$ does not belong to l_2.

Instead, let us take a particular $\Psi(\alpha) \in L_2[0, 2\pi]$ such that

$$\frac{1}{2\pi}\int_0^{2\pi} |\lambda(\alpha)|^{2p}|\Psi(\alpha)|^2 d\alpha \ge \max_\alpha\left(|\lambda(\alpha)|^{2p} - \varepsilon\right)\frac{1}{2\pi}\int_0^{2\pi} |\Psi(\alpha)|^2 d\alpha, \tag{10.102}$$

where $\varepsilon > 0$ is given. For an arbitrary ε, estimate (10.102) can always be guaranteed by choosing

$$\Psi(\alpha) = \begin{cases} 1, & \text{if} \quad \alpha \in [\alpha^* - \delta, \alpha^* + \delta], \\ 0, & \text{if} \quad \alpha \notin [\alpha^* - \delta, \alpha^* + \delta], \end{cases}$$

where $\alpha^* = \arg\max\limits_\alpha |\lambda(\alpha)|$ and $\delta > 0$. Indeed, as the function $|\lambda(\alpha)|^{2p}$ is continuous, inequality (10.102) will hold for a sufficiently small $\delta = \delta(\varepsilon)$.

If estimate (10.101) does not take place, then we can find a sequence h_k, $k = 0, 1, 2, 3, \ldots$, and the corresponding sequence $\tau_k = \tau(h_k)$ such that

$$c_k = \left(\max_{\alpha} |\lambda(\alpha, h_k)|\right)^{[T/\tau_k]} \longrightarrow \infty \quad \text{as} \quad k \longrightarrow \infty.$$

Let us set $\varepsilon = 1$ and choose $\Psi(\alpha)$ to satisfy (10.102). Define ψ_m as Fourier coefficients of the function $\Psi(\alpha)$, according to formula (10.98). Then, inequality (10.102) for $p_k = [T/\tau_k]$ transforms into:

$$\|u^{p_k}\|^2 \geq (c_k^2 - 1)\|\psi\|^2 \Longrightarrow \|u^{p_k}\| \geq (c_k - 1)\|\psi\|,$$
$$c_k \longrightarrow \infty \quad \text{as} \quad k \longrightarrow \infty,$$

i.e., there is indeed no stability (10.96) with respect to the initial data. $\square$

Theorem 10.3 establishes *equivalence* between the von Neumann spectral condition and the l_2 stability of scheme (10.95) with respect to the initial data. In fact, one can go even further and prove that the von Neumann spectral condition is *necessary and sufficient* for the full-fledged l_2 stability of the scheme (10.95) as well, i.e., when the right-hand side φ_m^p is not disregarded. One implication, the necessity, immediately follows from Theorem 10.3, because if the von Neumann condition does not hold, then the scheme is unstable even with respect to the initial data. The proof of the other implication, the sufficiency, can be found in [GR87, § 25]. This proof is based on using the discrete Green's functions. In general, once stability with respect to the initial data has been established, stability of the full inhomogeneous problem can be derived using the Duhamel principle. This principle basically says that the solution to the inhomogeneous problem can be obtained as linear superposition of the solutions to some specially chosen homogeneous problems. Consequently, a stability estimate for the inhomogeneous problem can be obtained on the basis of stability estimates for a series of homogeneous problems, see [Str04, Chapter 9].

10.3.6 Scalar Equations vs. Systems

As of yet, our analysis of finite-difference stability has focused mostly on scalar equations; we have considered a 2×2 system only in Examples 10 and 11 of Section 10.3.3. In Examples 5 and 9, we have also considered scalar difference equations that connect the values of the solution on more than two consecutive time levels; those can be reduced to systems on two time levels.

In general, a constant coefficient finite-difference Cauchy problem with vector unknowns (i.e., a system) can be written in the form similar to (10.95):

$$\sum_{j=-j_{\text{left}}}^{j_{\text{right}}} \boldsymbol{B}_j \boldsymbol{u}_{m+j}^{p+1} - \sum_{j=-j_{\text{left}}}^{j_{\text{right}}} \boldsymbol{A}_j \boldsymbol{u}_{m+j}^{p} = \boldsymbol{\varphi}_m^p, \tag{10.103}$$
$$\boldsymbol{u}_m^0 = \boldsymbol{\psi}_m, \quad m = 0, \pm 1, \pm 2, \ldots, \quad p = 0, 1, \ldots, [T/\tau] - 1,$$

under the assumption that the matrices

$$\sum_{j=-j_{\text{left}}}^{j_{\text{right}}} \boldsymbol{B}_j e^{i\alpha j}, \quad 0 \le \alpha < 2\pi,$$

are non-singular. In formula (10.103), $\boldsymbol{u}_m^p$, $\boldsymbol{\varphi}_m^p$, and $\boldsymbol{\psi}_m$ are grid vector functions of a fixed dimension, and $\boldsymbol{A}_j = \boldsymbol{A}_j(h)$, $\boldsymbol{B}_j = \boldsymbol{B}_j(h)$, $j = -j_{\text{left}}, \dots, j_{\text{right}}$, are given square matrices of the same dimension.

Solution to the homogeneous counterpart of equation (10.103) can be sought for in the form (10.80), where $\boldsymbol{u}^0 = \boldsymbol{u}^0(\alpha, h)$ and $\lambda = \lambda(\alpha, h)$ are the eigenvectors and eigenvalues, respectively, of the amplification matrix of scheme (10.103):

$$\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\alpha, h) = \left(\sum_{j=-j_{\text{left}}}^{j_{\text{right}}} \boldsymbol{B}_j e^{i\alpha j} \right)^{-1} \left(\sum_{j=-j_{\text{left}}}^{j_{\text{right}}} \boldsymbol{A}_j e^{i\alpha j} \right) \tag{10.104}$$

The von Neumann spectral condition (10.97) is still necessary for stability of systems in both C and l_2. Indeed, if it is not met, then estimate (10.101) won't hold, and the scheme will develop an exponential instability. This can be seen by applying the respective scalar argument to any component of the corresponding vector solution.

Yet the von Neumann condition remains *only a necessary* stability condition for systems in either C or l_2. For C, the analysis of sufficient conditions becomes cumbersome already for the scalar case (see Section 10.3.4), and even in l_2 obtaining sufficient conditions for systems proves rather involved.

Qualitatively, the difficulties stem from the fact that the amplification matrix (10.104) may have multiple eigenvalues and as a consequence, may not necessarily have a full set of eigenvectors. If a multiple eigenvalue occurs exactly on the unit circle or just outside the unit disk, this may still cause instability even when all the eigenvalues satisfy the von Neumann constraint (10.97) (similar to Theorem 9.2).

An example is provided by the leap-frog scheme (10.86). If $r = 1$ and $\alpha = \pi/2$, then we have $\lambda_{1,2} = i$, and if $r = 1$ and $\alpha = 3\pi/2$, then $\lambda_{1,2} = -i$. In either case, in addition to (10.80) there will be a solution of the form $\boldsymbol{u}_m^p = p\lambda^p \left[\boldsymbol{u}^0 e^{i\alpha m}\right]$, which is a manifestation of a gradually (linearly) developing instability.

Of course, if the amplification matrix appears normal (a matrix that commutes with its adjoint) and therefore unitarily diagonalizable, then none of the aforementioned difficulties is present, and the von Neumann condition becomes not only necessary but also sufficient for stability of the vector scheme (10.103) in l_2.

Otherwise, the question of stability for scheme (10.103) can be *equivalently reformulated* using the new concept of stability for families of matrices. A family of square matrices (of a given fixed dimension) is said to be stable if there is a constant $K > 0$ such that for any particular matrix $\boldsymbol{\Lambda}$ from the family, and any positive integer p, the following estimate holds: $\|\boldsymbol{\Lambda}^p\| \le K$. Scheme (10.103) is stable in l_2 if and only if the family of amplification matrices $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\alpha, h)$ given by (10.104) is stable in the sense of the previous definition (this family is parameterized by $\alpha \in [0, 2\pi)$ and $h > 0$). Theorem 10.4, known as the Kreiss matrix theorem, provides some necessary and sufficient conditions for a family of matrices to be stable.

THEOREM 10.4 (Kreiss)
Stability of a family of matrices $\mathbf{\Lambda}$ is equivalent to any of the following:

1. *There is a constant $C_1 > 0$ such that for any matrix $\mathbf{\Lambda}$ from the given family, and any complex number z, $|z| > 1$, there is a resolvent $(\mathbf{\Lambda} - z\boldsymbol{I})^{-1}$ bounded as:*
$$\left\|(\mathbf{\Lambda} - z\boldsymbol{I})^{-1}\right\| \leq \frac{C_1}{|z|-1}.$$
2. *There are constants $C_2 > 0$ and $C_3 > 0$, and for any matrix $\mathbf{\Lambda}$ from the given family there is a non-singular matrix $\boldsymbol{M}$ such that $\|\boldsymbol{M}\| \leq C_2$, $\|\boldsymbol{M}^{-1}\| \leq C_2$, and the matrix $\boldsymbol{D} \stackrel{\text{def}}{=} \boldsymbol{M}\mathbf{\Lambda}\boldsymbol{M}^{-1}$ is upper triangular, with the off-diagonal entries that satisfy:*
$$|d_{ij}| \leq C_3 \min\{1-\kappa_i, 1-\kappa_j\},$$
where $\kappa_i = d_{ii}$ and $\kappa_j = d_{jj}$ are the corresponding diagonal entries of $\boldsymbol{D}$, i.e., the eigenvalues of $\mathbf{\Lambda}$.
3. *There is a constant $C_4 > 0$, and for any matrix $\mathbf{\Lambda}$ from the given family there is a Hermitian positive definite matrix $\boldsymbol{H}$, such that*
$$C_4^{-1}\boldsymbol{I} \leq \boldsymbol{H} \leq C_4\boldsymbol{I} \quad \text{and} \quad \mathbf{\Lambda}^*\boldsymbol{H}\mathbf{\Lambda} \leq \boldsymbol{H}.$$

The proof can be found in [RM67, Chapter 4] or [Str04, Chapter 9].

Exercises

1. Consider the so-called weighted scheme for the heat equation:
$$\frac{u_m^{p+1} - u_m^p}{\tau} = \sigma\frac{u_{m+1}^{p+1} - 2u_m^{p+1} + u_{m-1}^{p+1}}{h^2} + (1-\sigma)\frac{u_{m+1}^p - 2u_m^p + u_{m-1}^p}{h^2},$$
$$u_m^0 = \psi_m, \quad m = 0, \pm 1, \pm 2, \dots, \quad p = 0, 1, \dots, [T/\tau] - 1,$$
where the real parameter $\sigma \in [0,1]$ is called the weight (between the fully explicit scheme, $\sigma = 0$, and fully implicit scheme, $\sigma = 1$). What values of σ guarantee that the scheme will meet the von Neumann stability condition for any $r = \tau/h^2 = \text{const}$?
2. Consider the Cauchy problem (10.87) for the heat equation. The scheme:
$$\frac{u_m^{p+1} - u_m^{p-1}}{2\tau} - a^2\frac{u_{m+1}^p - 2u_m^p + u_{m-1}^p}{h^2} = \varphi_m^p,$$
$$u_m^0 = \psi_m, \quad u_m^1 = \tilde{\psi}_m, \quad m = 0, \pm 1, \pm 2, \dots, \quad p = 0, 1, \dots, [T/\tau] - 1,$$
where we assume that $\varphi(x,0) \equiv 0$ and define:
$$\tilde{\psi}_m = u(x,0) + \tau\frac{\partial u(x,0)}{\partial t}\Big|_{x=x_m} = u(x,0) + \tau a^2\frac{\partial^2 u(x,0)}{\partial x^2}\Big|_{x=x_m} = \psi_m + \tau a^2\psi''(x_m),$$
approximates problem (10.87) on its smooth solutions with accuracy $\mathscr{O}(\tau^2 + h^2)$. Does this scheme satisfy the von Neumann spectral stability condition for $r = \tau/h^2 = \text{const}$?

3. For the two-dimensional Cauchy problem:

$$\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = \varphi(x,y,t), \quad -\infty < x, y < \infty, \quad 0 < t \leq T,$$
$$u(x,y,0) = \psi(x,y), \quad -\infty < x, y < \infty,$$

investigate the von Neumann spectral stability of:

   a) The first order explicit scheme:

$$\frac{u_{m,n}^{p+1} - u_{m,n}^{p}}{\tau} - \frac{u_{m+1,n}^{p} - u_{m,n}^{p}}{h} - \frac{u_{m,n+1}^{p} - u_{m,n}^{p}}{h} = \varphi_{m,n}^{p},$$
$$u_{m,n}^{0} = \psi_{m,n}, \quad m, n = 0, \pm1, \pm2, \ldots, \quad p = 0, 1, \ldots, [T/\tau] - 1;$$

   b) The second order explicit scheme:

$$\frac{u_{m,n}^{p+1} - u_{m,n}^{p-1}}{2\tau} - \frac{u_{m+1,n}^{p} - u_{m-1,n}^{p}}{2h} - \frac{u_{m,n+1}^{p} - u_{m,n-1}^{p}}{2h} = \varphi_{m,n}^{p},$$
$$u_{m,n}^{0} = \psi_{m,n}, \quad u_{m,n}^{1} = \psi_{m,n} + \tau[\psi_x'(x_m, y_n) + \psi_y'(x_m, y_n) + \varphi(x_m, y_n, 0)],$$
$$m, n = 0, \pm1, \pm2, \ldots, \quad p = 0, 1, \ldots, [T/\tau] - 1.$$

4. Investigate the von Neumann spectral stability of the implicit two-dimensional scheme for the homogeneous heat equation:

$$\frac{u_{m,n}^{p+1} - u_{m,n}^{p}}{\tau} = \frac{u_{m+1,n}^{p+1} - 2u_{m,n}^{p+1} + u_{m-1,n}^{p+1}}{h^2} + \frac{u_{m,n+1}^{p+1} - 2u_{m,n}^{p+1} + u_{m,n-1}^{p+1}}{h^2}$$
$$u_{m,n}^{0} = \psi_{m,n}, \quad m, n = 0, \pm1, \pm2, \ldots, \quad p = 0, 1, \ldots, [T/\tau] - 1.$$

5. Investigate the von Neumann stability of the implicit upwind scheme for the Cauchy problem (10.82):

$$\begin{aligned} &\frac{u_m^{p+1} - u_m^p}{\tau} - \frac{u_{m+1}^{p+1} - u_m^{p+1}}{h} = \varphi_m^p, \\ &u_m^0 = \psi_m, \quad m = 0, \pm1, \pm2, \ldots, \quad p = 0, 1, \ldots, [T/\tau] - 1. \end{aligned} \tag{10.105}$$

6. Investigate the von Neumann stability of the implicit downwind scheme for the Cauchy problem (10.82):

$$\begin{aligned} &\frac{u_m^{p+1} - u_m^p}{\tau} - \frac{u_m^{p+1} - u_{m-1}^{p+1}}{h} = \varphi_m^p, \\ &u_m^0 = \psi_m, \quad m = 0, \pm1, \pm2, \ldots, \quad p = 0, 1, \ldots, [T/\tau] - 1. \end{aligned} \tag{10.106}$$

7. Investigate the von Neumann stability of the implicit central scheme for the Cauchy problem (10.82):

$$\begin{aligned} &\frac{u_m^{p+1} - u_m^p}{\tau} - \frac{u_{m+1}^{p+1} - u_{m-1}^{p+1}}{2h} = \varphi_m^p, \\ &u_m^0 = \psi_m, \quad m = 0, \pm1, \pm2, \ldots, \quad p = 0, 1, \ldots, [T/\tau] - 1. \end{aligned} \tag{10.107}$$

8.* Transform the leap-frog scheme (10.86) of Example 5, Section 10.3.3, and the central-difference scheme for the d'Alembert equation of Example 9, Section 10.3.3, to the schemes written for finite-difference systems, as opposed to scalar equations, but connecting only two, as opposed to three, consecutive time levels of the grid. Investigate the von Neumann stability by calculating spectra of the corresponding amplification matrices (10.104).

Hint. Use the difference $\{u_m^{p+1} - u_m^p\}$ as the second unknown grid function.

10.4 Stability for Problems with Variable Coefficients

The von Neumann necessary condition that we have introduced in Section 10.3 to analyze stability of linear finite-difference Cauchy problems with constant coefficients can, in fact, be applied to a wider class of formulations. A simple extension that we describe in this section allows one to exploit the von Neumann condition to analyze stability of problems with variable coefficients (continuous, but not necessarily constant) and even some nonlinear problems.

10.4.1 The Principle of Frozen Coefficients

Introduce a uniform Cartesian grid: $x_m = mh$, $m = 0, \pm 1, \pm 2, \dots$, $t_p = p\tau$, $p = 0, 1, 2, \dots$, and consider a finite-difference Cauchy problem for the homogeneous heat equation with the variable coefficient of heat conduction $a = a(x,t)$:

$$\begin{gathered} \frac{u_m^{p+1} - u_m^p}{\tau} - a(x_m, t_p)\frac{u_{m+1}^p - 2u_m^p + u_{m-1}^p}{h^2} = 0, \\ u_m^0 = \psi(x_m), \quad m = 0, \pm 1, \pm 2, \dots, \quad p \geq 0. \end{gathered} \tag{10.108}$$

Next, take an arbitrary point $(\tilde{x}, \tilde{t})$ in the domain of problem (10.108) and "freeze" the coefficients of problem (10.108) at this point. Then, we arrive at the constant-coefficient finite-difference equation:

$$\begin{gathered} \frac{u_m^{p+1} - u_m^p}{\tau} - a(\tilde{x}, \tilde{t})\frac{u_{m+1}^p - 2u_m^p + u_{m-1}^p}{h^2} = 0, \\ m = 0, \pm 1, \pm 2, \dots, \quad p \geq 0. \end{gathered} \tag{10.109}$$

Having obtained equation (10.109), we can formulate the following principle of frozen coefficients. *For the original variable-coefficient Cauchy problem (10.108) to be stable it is necessary that the constant-coefficient Cauchy problem for the difference equation (10.109) satisfies the von Neumann spectral stability condition.*

To justify the principle of frozen coefficients, we will provide an heuristic argument rather than a proof. When the grid is refined, the variation of the coefficient $a(x,t)$ in a neighborhood of the point $(\tilde{x}, \tilde{t})$ becomes smaller if measured over any finite fixed number of grid cells that have size h in space and size τ in time. This is true because of the continuity of the function $a = a(x,t)$. In other words, the finer the grid, the closer is $a(x,t)$ to $a(\tilde{x},\tilde{t})$ as long as (x,t) is no more than so many cells away from $(\tilde{x},\tilde{t})$. Consequently, if we were to perturb the solution of problem (10.108) on a fine grid at the moment of time $t = \tilde{t}$ near the space location $x = \tilde{x}$, then over short time intervals these perturbations would have evolved pretty much the as if they were perturbations of the solution to the constant-coefficient equation (10.109).

It is clear that the previous argument is quite general. It is not affected by the number of space dimensions, the number of unknown functions, or the specific type of the finite-difference equation or system.

In Section 10.3.3, we analyzed a Cauchy problem for the equation of type (10.109), see Example 6, and found that for the von Neumann stability condition to hold the ratio $r = \tau/h^2$ must satisfy the inequality:

$$r \leq \frac{1}{2a(\tilde{x},\tilde{t})}. \tag{10.110}$$

According to the principle of frozen coefficients, stability of scheme (10.108) requires that condition (10.110) be met for any $(\tilde{x},\tilde{t})$. Therefore, altogether the ratio $r = \tau/h^2$ must satisfy the inequality:

$$r \leq \frac{1}{2\max\limits_{x,t} a(x,t)}. \tag{10.111}$$

The principle of frozen coefficients can also provide an heuristic argument for the analysis of stability of nonlinear difference equations. We illustrate this using an example of a Cauchy problem for the nonlinear heat equation:

$$\begin{gathered}\frac{\partial u}{\partial t} - (1+u^2)\frac{\partial^2 u}{\partial x^2} = 0, \quad -\infty < x < \infty, \quad 0 < t \leq T, \\ u(x,0) = \psi(x), \quad -\infty < x < \infty.\end{gathered}$$

We approximate this problem by means of an explicit scheme:

$$\begin{gathered}\frac{u_m^{p+1} - u_m^p}{\tau_p} - \left(1 + (u_m^p)^2\right)\frac{u_{m+1}^p - 2u_m^p + u_{m-1}^p}{h^2} = 0, \\ u_m^0 = \psi(x_m), \quad m = 0, \pm 1, \pm 2, \dots, \quad p = 0, 1, 2, \dots.\end{gathered} \tag{10.112}$$

The scheme is built on a uniform spatial grid: $x_m = mh$, $m = 0, \pm 1, \pm 2, \dots$, but with a non-uniform temporal grid, such that the size $\tau_p = t_{p+1} - t_p$ may vary from one time level to another. The finite-difference solution can still be obtained by marching.

Assume that we have already marched all the way up the time level t_p and computed the solution u_m^p, $m = 0, \pm 1, \pm 2, \dots$. To continue marching, we first need to select the next grid size τ_p. This can be done by interpreting the finite-difference equation to be solved at $t = t_p$ with respect to u_m^{p+1} as a linear equation:

$$\frac{u_m^{p+1} - u_m^p}{\tau_p} - a_m^p \frac{u_{m+1}^p - 2u_m^p + u_{m-1}^p}{h^2} = 0 \tag{10.113}$$

with the given variable coefficient of heat conduction: $a_m^p \equiv 1+(u_m^p)^2$. Indeed, it is natural to think that the values of the grid function u_m^p are close to the values $u(x_m,t_p)$ of the continuous solution $u(x,t)$. Then, the discrete heat conduction coefficient a_m^p will be close to the projection $a(x_m,t_p)$ of the continuous function $a(x,t) = 1+u(x,t)^2$ onto the grid. This function may vary only slightly over a few temporal steps.

By applying the principle of frozen coefficients to equation (10.113), we arrive at the constraint (10.111) for the grid sizes that is necessary for stability:

$$\frac{\tau_p}{h^2} = r_p \leq \frac{1}{2\max\limits_m a_m^p} = \frac{1}{2\max\limits_m(1+(u_m^p)^2)}.$$

Consequently, when marching equation (10.112) one should select the temporal grid size τ_p for each $p = 0,1,2,\dots$ based on the inequality:

$$\tau_p \leq \frac{h^2}{2\max\limits_m(1+(u_m^p)^2)}.$$

Numerical experiments corroborate correctness of these heuristic arguments.

If stability condition obtained by considering the Cauchy problem with frozen coefficients (at an arbitrary point of the domain) is violated, then we expect that there will be no stability. We re-emphasize though that our justification for the principle of frozen coefficients was heuristic rather than rigorous. There are, in fact, counter-examples, when the problem with variable coefficients is stable, whereas the problems with frozen coefficients are unstable. Those counter-examples, however, are fairly "exotic" in nature, like the one given by Strang in [Str66] that involves a second order differential equation with complex coefficients: $u_t = i[(\sin x)u_x]_x$. At the same time, the analysis of [Str66] shows that for some important classes of differential equations/systems that include, in particular, all first order systems (e.g., hyperbolic), the principle of frozen coefficients holds in the sense of l_2 for explicit finite-difference schemes. The same result is also known to be true for parabolic equations, see [RM67, Chapter 5]. Therefore, for all practical purposes hereafter, we will still regard the principle of frozen coefficients as a necessary condition for stability. Moreover, in Section 10.6.1 we will use the maximum principle and show that the explicit finite-difference scheme for a variable coefficient heat equation is actually stable if it satisfies the condition based on the principle of frozen coefficients.

We should also mention that the principle of frozen coefficients as formulated in this section applies only to Cauchy problems. For an initial boundary value problem formulated on a finite interval, the foregoing analysis is not sufficient. Even if the necessary stability condition based on the principle of frozen coefficients holds, the overall problem on the finite interval can still be either stable or unstable depending on the choice of the boundary conditions at the endpoints of the interval. In Section 10.5.1, we discuss the Babenko-Gelfand stability criterion that takes into account the effect of boundary conditions in the case of a problem on an interval.