## Example 8

The von Neumann analysis also applies when studying stability of a scheme in the case of more than one spatial variable. Consider, for instance, a Cauchy problem for the heat equation on the $(x, y)$ plane:

$$
\begin{gathered}
\frac{\partial u}{\partial t}=\frac{\partial^{2} u}{\partial x^{2}}+\frac{\partial^{2} u}{\partial y^{2}} \quad-\infty<x, y<\infty, 0<t \leq T \\
u(x, y, 0)=\psi(x, y), \quad-\infty<x, y<\infty
\end{gathered}
$$

We approximate this problem on the uniform Cartesian grid: $\left(x_{m}, y_{n}, t_{p}\right)=$ ( $m h, n h, p \tau$ ). Replacing the derivatives with difference quotients we obtain:

$$
\begin{gathered}
\frac{u_{m, n}^{p+1}-u_{m, n}^{p}}{\tau}=\frac{u_{m+1, n}^{p}-2 u_{m, n}^{p}+u_{m-1, n}^{p}}{h^{2}}+\frac{u_{m, n+1}^{p}-2 u_{m, n}^{p}+u_{m, n-1}^{p}}{h^{2}} \\
u_{m, n}^{0}=\psi_{m, n}, \quad m, n=0, \pm 1, \pm 2, \ldots, \quad p=0,1, \ldots,[T / \tau]-1 .
\end{gathered}
$$

The resulting scheme is explicit. To analyze its stability, we specify $u_{m, n}^{0}$ in the form of a two-dimensional harmonic $e^{i(\alpha m+\beta n)}$ determined by two real parameters $\alpha$ and $\beta$, and generate a solution in the form:

$$
u_{m, n}^{p}=\lambda^{p}(\alpha, \beta) e^{i(\alpha m+\beta n)}
$$

Substituting this expression into the homogeneous difference equation, we find after some easy equivalence transformations:

$$
\lambda(\alpha, \beta)=1-4 r\left(\sin ^{2} \frac{\alpha}{2}+\sin ^{2} \frac{\beta}{2}\right)
$$

When the real quantities $\alpha$ and $\beta$ vary between 0 and $2 \pi$, the point $\lambda=\lambda(\alpha, \beta)$ sweeps the interval $1-8 r \leq \lambda \leq 1$ of the real axis. The von Neumann stability condition is satisfied if $1-8 r \geq-1$, i.e., when $r \leq 1 / 4$.

## Example 9

In addition to Example 5, let us now consider another example of the scheme that connects the values of the difference solution on three, rather than two, consecutive time levels of the grid.

A Cauchy problem for the one-dimensional homogeneous d'Alembert (wave) equation:

$$
\begin{gathered}
\frac{\partial^{2} u}{\partial t^{2}}-\frac{\partial^{2} u}{\partial x^{2}}=0, \quad-\infty<x<\infty, \quad 0<t \leq T \\
u(x, 0)=\psi^{(0)}(x), \quad \frac{\partial u(x, 0)}{\partial t}=\psi^{(1)}(x), \quad-\infty<x<\infty
\end{gathered}
$$

can be approximated on a uniform Cartesian grid by the following scheme:

$$
\begin{gathered}
\frac{u_{m}^{p+1}-2 u_{m}^{p}+u_{m}^{p-1}}{\tau^{2}}-\frac{u_{m+1}^{p}-2 u_{m}^{p}+u_{m-1}^{p}}{h^{2}}=0 \\
u_{m}^{0}=\psi_{m}^{(0)}, \quad \frac{u_{m}^{1}-u_{m}^{0}}{\tau}=\psi_{m}^{(1)} \\
m=0, \pm 1, \pm 2, \ldots, \quad p=1,2, \ldots,[T / \tau]-1
\end{gathered}
$$

Substituting a solution of type (10.76) into the foregoing finite-difference equation, we obtain the following quadratic equation for determining $\lambda=\lambda(\alpha)$ :

$$
\lambda^{2}-2\left(1-2 r^{2} \sin ^{2} \frac{\alpha}{2}\right) \lambda+1=0, \quad r=\frac{\tau}{h}
$$

The product of the two roots of this equation is equal to one. If its discriminant:

$$
D(\alpha)=4 r^{2} \sin ^{2} \frac{\alpha}{2}\left(r^{2} \sin ^{2} \frac{\alpha}{2}-1\right)
$$

is negative, then the roots $\lambda_{1}(\alpha)$ and $\lambda_{2}(\alpha)$ are complex conjugate and both have a unit modulus. If $r<1$, the discriminant $D(\alpha)$ remains negative for all $\alpha \in[0,2 \pi)$.

(a) $r<1$

(b) $r>1$

FIGURE 10.9: Spectrum of the scheme for the wave equation.

The spectrum for this case is shown in Figure 10.9(a); it fills an arc of the unit circle. If $r=1$, the spectrum fills exactly the entire unit circle. When $r>1$, the discriminant $D(\alpha)$ may be either negative or positive depending on the value of $\alpha$. In this case, once the argument $\alpha$ increases from 0 to $\pi$ the roots $\lambda_{1}(\alpha)$ and $\lambda_{2}(\alpha)$ depart from the point $\lambda=1$ and move along the unit circle: One root moves clockwise and the other root counterclockwise, and then they merge at $\lambda=-1$. After that one root moves away from this point along the real axis to the right, and the other one to the left, because they are both real and their product $\lambda_{1} \lambda_{2}=1$, see Figure 10.9(b). The von Neumann stability condition is met for $r \leq 1$.

