Then, the von Neumann condition (10.81) in the form $|\lambda(\alpha)| \leq 1 + c_1\tau$ is satisfied for $c_1 = r/2$. It is clear that the requirement $\tau = rh^2$ puts a considerably stricter constraint on the rate of decay of the temporal grid size τ as $h \longrightarrow 0$ than the previous requirement $\tau = rh$ does. Still, that previous requirement was sufficient for the von Neumann condition to hold for the difference schemes (10.69)–(10.70) and (10.83) that approximate the same Cauchy problem (10.82).

We also note that the Courant, Friedrichs, and Lewy condition of Section 10.1.4 allows us to claim that the scheme (10.84) is unstable only for $\tau/h = r > 1$, but does not allow us to judge the stability for $\tau/h = r \leq 1$. As such, it appears weaker than the von Neumann condition.

Example 4

The instability of scheme (10.84) for $\tau/h = r = \text{const}$ can be fixed by changing the way the time derivative is approximated. Instead of (10.84), consider the scheme:

$$\begin{aligned} \frac{u_m^{p+1} - (u_{m-1}^p + u_{m+1}^p)/2}{\tau} - \frac{u_{m+1}^p - u_{m-1}^p}{2h} &= \varphi_m^p, \\ u_m^0 &= \psi_m, \end{aligned} \tag{10.85}$$

obtained by replacing u_m^p with $(u_{m-1}^p + u_{m+1}^p)/2$. This is, in fact, a general approach attributed to Friedrichs, and the scheme (10.85) is known as the Lax-Friedrichs scheme; we have first introduced it in Section 10.2.4. The equation to determine the spectrum for the scheme (10.85) reads:

$$\frac{\lambda - (e^{i\alpha} + e^{-i\alpha})/2}{\tau} - \frac{e^{i\alpha} - e^{-i\alpha}}{2h} = 0,$$

which yields:

$$\frac{\lambda - \cos\alpha}{\tau} - \frac{i\sin\alpha}{h} = 0$$

and

$$\lambda(\alpha) = \cos\alpha + ir\sin\alpha,$$

where $r = \tau/h = \text{const}$. Consequently,

$$|\lambda(\alpha)|^2 = \cos^2\alpha + r^2\sin^2\alpha.$$

Clearly, the von Neumann condition (10.79) is satisfied for $r \leq 1$, because then $|\lambda|^2 \leq \cos^2\alpha + \sin^2\alpha = 1$. For $r > 1$, the von Neumann condition is violated.

Example 5

Finally, consider the leap-frog scheme (10.36) for problem (10.82). The corresponding homogeneous finite-difference equation is written as:

$$\frac{u_m^{p+1} - u_m^{p-1}}{2\tau} - \frac{u_{m+1}^p - u_{m-1}^p}{2h} = 0, \tag{10.86}$$

and for the spectrum we obtain:

$$\frac{\lambda - \lambda^{-1}}{2\tau} - \frac{e^{i\alpha} - e^{-i\alpha}}{2h} = 0.$$

This is a quadratic equation with respect to λ:

$$\lambda^2 - i2r\lambda \sin\alpha - 1 = 0,$$

where $r = \tau/h = \text{const}$. The roots of this equation are given by:

$$\lambda_{1,2} = ir\sin\alpha \pm \sqrt{1 - r^2\sin^2\alpha}.$$

We notice that when $r \leq 1$, then $|\lambda_{1,2}|^2 = r^2\sin^2\alpha + (1 - r^2\sin^2\alpha) = 1$, so that the entire spectrum lies precisely on the unit circle and the von Neumann condition (10.79) is satisfied. Otherwise, when $r > 1$, we again take $\alpha = \pi/2$ and obtain: $|\lambda_1| = |ir + i\sqrt{r^2-1}| = r + \sqrt{r^2-1} > 1$, which means that the von Neumann condition is not met. We thus see that the von Neumann criterion can be applied to a finite-difference equation that connects more than two consecutive time levels on the grid. However, for $r = 1$ the two-step scheme (10.86) proves unstable, see Section 10.3.6.

Next, we will consider two schemes that approximate the following Cauchy problem for the heat equation:

$$\begin{aligned} \frac{\partial u}{\partial t} - a^2\frac{\partial^2 u}{\partial x^2} &= \varphi(x,t), \qquad -\infty < x < \infty,\ \ 0 < t \leq T, \\ u(x,0) &= \psi(x), \qquad -\infty < x < \infty. \end{aligned} \tag{10.87}$$

Example 6

The first scheme is explicit:

$$\frac{u_m^{p+1} - u_m^p}{\tau} - a^2\frac{u_{m+1}^p - 2u_m^p + u_{m-1}^p}{h^2} = \varphi_m^p,$$
$$u_m^0 = \psi_m, \quad m = 0, \pm1, \pm2, \dots, \quad p = 0, 1, \dots, [T/\tau] - 1.$$

It allows us to compute $\{u_m^{p+1}\}$ in the closed form via $\{u_m^p\}$:

$$u_m^{p+1} = (1 - 2ra^2)u_m^p + ra^2(u_{m+1}^p + u_{m-1}^p) + \tau\varphi_m^p, \quad p = 0, 1, \dots, [T/\tau] - 1,$$

where $r = \tau/h^2 = \text{const}$. Substitution of $u_m^p = \lambda^p e^{i\alpha m}$ into the corresponding homogeneous difference equation yields:

$$\frac{\lambda - 1}{\tau} - a^2\frac{e^{-i\alpha} - 2 + e^{i\alpha}}{h^2} = 0.$$

By noticing that

$$\frac{e^{-i\alpha} - 2 + e^{i\alpha}}{4} = -\sin^2\frac{\alpha}{2},$$