7. Consider Cauchy problem (10.67) for the heat equation, and approximate it with the predictor-corrector scheme designed as follows. The auxiliary grid function $\tilde{u}_m^{p+1/2}$ is to be computed by the implicit method:

$$\frac{\tilde{u}_m^{p+1/2}-u_m^p}{\tau/2}-\frac{\tilde{u}_{m+1}^{p+1/2}-2\tilde{u}_m^{p+1/2}+\tilde{u}_{m-1}^{p+1/2}}{h^2}=0, \quad m=0,\pm 1,\pm 2,\dots,$$

and the actual solution u_m^{p+1} at $t=t_{p+1}$ is to be computed by the scheme:

$$\frac{u_m^{p+1}-u_m^p}{\tau}-\frac{\tilde{u}_{m+1}^{p+1/2}-2\tilde{u}_m^{p+1/2}+\tilde{u}_{m-1}^{p+1/2}}{h^2}=0, \quad u_m^0=\psi(mh).$$

Prove that the overall predictor-corrector scheme has accuracy $\mathscr{O}(\tau^2+h^2)$ on the smooth solution $u=u(x,t)$.

8. Consider a modified scheme $\boldsymbol{L}_h u^{(h)}=f^{(h)}$ of type (10.47) (modification of the right-hand side only):

$$\boldsymbol{L}_h u^{(h)}=\begin{cases} a^0 u_m^{p+1}+a_{-1}u_{m-1}^p+a_0u_m^p+a_1u_{m+1}^p, \\ u_m^0, \end{cases}$$

$$f^{(h)}=\begin{cases} \varphi(x_m,t_p)+\dfrac{rh}{2}(\varphi_t+\varphi_x)\Big|_{(x_m,t_p)}, \\ \psi(x_m), \end{cases}$$

and define the coefficients of the operator $\boldsymbol{L}_h$ according to formula (10.55) that corresponds to the Lax-Wendroff method. Show that the resulting scheme approximates problem (10.8) with second order accuracy for an arbitrary sufficiently smooth right-hand side $\varphi(x,t)$ (not necessarily zero).

9. Represent scheme (10.47) with $\varphi(x,t)=0$ in the form:

$$u_m^{p+1}=b_{-1}u_{m-1}^p+b_0u_m^p+b_1u_{m+1}^p, \tag{10.68}$$

where $b_{-1}=-a_{-1}/a^0$, $b_0=-a_0/a^0$, and $b_1=-a_1/a^0$, see Section 10.2.3. Scheme (10.68) is said to be *monotone* if $b_j\geq 0,\ j=-1,0,1$. Adopting the terminology of Section 10.2.3, let the primary constraint be the order of accuracy of at least $\mathscr{O}(h)$, and the secondary constraint be monotonicity of the scheme. In the three-dimensional space of vectors $\{b_{-1},b_0,b_1\}$, describe the set $M_0=M_{\text{pr}}\cap M_{\text{sec}}$ (in this case, the distance between the sets M_{pr} and M_{sec} is zero).

Answer. If $r=\tau/h>1$, then $M_0=\emptyset$. If $r=1$, then $M_0=\{(0,0,1)\}$. If $0<r<1$, then M_0 is the interval with the endpoints: $\left(\frac{1-r}{2},0,\frac{1+r}{2}\right)$ and $(0,1-r,r)$.

10. Prove that the monotone schemes defined the same way as in Exercise 9 for $\varphi=0$ and $j=-j_{\text{left}},\dots,j_{\text{right}}$ satisfy the maximum principle from page 315, i.e., that the maximum of the corresponding difference solution will not increase as the time elapses.

11.* **(Godunov theorem)** Prove that no one-step explicit monotone scheme may have accuracy higher than $\mathscr{O}(h)$ on smooth solutions of the differential equation $u_t+cu_x=0$. **Hint.** One version of the proof, due to Harten, Hyman, and Lax, can be found, e.g., in [Str04, pages 71–72]. Note also that there are exceptions, but they are all trivial. For example, the first order upwind scheme with $r=1$ has zero error on the exact solution $u=u(x-t)$ of $u_t+u_x=0$.

12. Consider a Cauchy problem for the one-dimensional wave (d'Alembert) equation:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = \varphi(x,t), \quad -\infty < x < \infty, \quad 0 < t \le T,$$
$$u(x,0) = \psi^{(0)}(x), \quad \frac{\partial u(x,0)}{\partial t} = \psi^{(1)}(x), \quad -\infty < x < \infty.$$

Analyze the approximation properties of the scheme $\boldsymbol{L}_h u^{(h)} = f^{(h)}$ on a smooth solution $u = u(x,t)$, where:

$$\boldsymbol{L}_h u^{(h)} = \begin{cases} \dfrac{u_m^{p+1} - 2u_m^p + u_m^{p-1}}{\tau^2} - \dfrac{u_{m+1}^p - 2u_m^p + u_{m-1}^p}{h^2}, \\ u_m^0, \\ \dfrac{u_m^1 - u_m^0}{\tau}, \end{cases}$$

$$f^{(h)} = \begin{cases} \varphi(x_m, t_p) \equiv \varphi_m^p, \\ \psi^{(0)}(x_m) \equiv \psi_m^{(0)}, \\ \psi^{(1)}(x_m) \equiv \psi_m^{(1)}. \end{cases}$$

Define the norm of $f^{(h)}$ as $\|f^{(h)}\|_{F_h} = \max\left\{\max\limits_p \sup\limits_m |\varphi_m^p|, \sup\limits_m |\psi_m^{(0)}|, \sup\limits_m |\psi_m^{(1)}|\right\}$ and show that when $\tau = rh$, $r = \text{const}$, the accuracy of the scheme is $\mathscr{O}(h)$.

Find an alternative way of specifying $\psi_m^{(1)}$ [instead of $\psi^{(1)}(x_m)$] so that to improve the order of accuracy and make it $\mathscr{O}(h^2)$. Note that $\varphi(x,t)$, $\psi^{(0)}(x)$, and $\psi^{(1)}(x)$ are given.

13. Consider the process of heat transfer on a finite interval, as opposed to the infinite line. It is governed by the heat equation subject to both initial and boundary conditions, which altogether yield the initial boundary value problem:

$$\begin{aligned} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} &= \varphi(x,t), && 0 < x < 1, \quad 0 < t \le T, \\ u(x,0) &= \psi^{(0)}(x), && 0 < x < 1, \\ u(0,t) &= \psi^{(1)}(t), && 0 < t \le T, \\ u(1,t) &= \psi^{(2)}(t), && 0 < t \le T. \end{aligned}$$

Use the grid: $x_m = mh$, $t_p = p\tau$, and approximate this problem with the scheme:

$$\frac{u_m^{p+1} - u_m^p}{\tau} - \frac{u_{m+1}^p - 2u_m^p + u_{m-1}^p}{h^2} = \varphi(x_m, t_p),$$
$$m = 1, 2, \dots, M-1, \quad h = 1/M, \quad p = 0, 1, \dots, [T/\tau] - 1,$$
$$u_m^0 = \psi^{(0)}(x_m), \quad m = 0, 1, \dots, M,$$
$$u_0^p = \psi^{(1)}(t_p), \quad p = 0, 1, \dots, [T/\tau],$$
$$u_M^p = \psi^{(2)}(t_p), \quad p = 0, 1, \dots, [T/\tau].$$

Define the norm in the space F_h as:

$$\|f^{(h)}\|_{F_h} = \max\left\{\max_{m,p} |\varphi(x_m, t_p)|, \max_m |\psi^{(0)}(x_m)|, \max_p |\psi^{(1)}(t_p)|, \max_p |\psi^{(2)}(t_p)|\right\},$$

and show that when $\tau = rh^2$, $r = \text{const}$, the accuracy of the scheme is $\mathscr{O}(h^2)$.