The quantity $\delta\phi$ is often referred to as the phase error.

In many applications related to the propagation of waves (for example, in computational aeroacoustics) it may be highly desirable to minimize the phase error of the scheme. Therefore, we can formulate the following problem: Among the schemes (10.57) that have a prescribed order of accuracy on a fixed stencil (primary constraint that defines the set M_{pr}), find the one that would have the minimum phase error on the largest possible subinterval $0 < k \leq k_0$ of the overall range of wavenumbers $0 < k \leq 2\pi/h$. Solving this problem would imply analyzing the dispersion relations similar to (10.58), but obtained for the general class of schemes (10.57).

The secondary constraint in this case can be taken as a requirement that the phase error be equal to zero. It can, however, be shown that only a small number of schemes (10.57) possess this property and only for the Courant number $cr = 1$. Consequently, for the schemes with the Courant number less than one (which would always be the case for multiple space dimensions) the primary and secondary constraints will indeed be incompatible, i.e., $M_{\text{pr}} \cap M_{\text{sec}} = \emptyset$. Then, finding the scheme with the minimum phase error on M_{pr} may require solving a fairly non-trivial optimization problem. Some particular formulations of this type have been studied in [TW93].

To provide another example of the phase error analysis, let us consider the Lax-Wendroff scheme [cf. formula (10.56)]:

$$\frac{u_m^{p+1} - u_m^p}{\tau} - c\frac{u_{m+1}^p - u_{m-1}^p}{2h} - c^2\frac{\tau}{2}\frac{u_{m+1}^p - 2u_m^p + u_{m-1}^p}{h^2} = 0.$$

Denoting $r = \tau/h = \text{const}$ as before, we obtain the amplification factor:

$$\lambda = 1 + icr\sin(kh) - 2(cr)^2\sin^2\frac{kh}{2} = |\lambda|e^{i\phi}.$$

Then, for the phase ϕ we have:

$$\tan\phi = \frac{cr\sin(kh)}{1 - 2(cr)^2\sin^2(kh/2)},$$

and similarly to the case of the first order upwind scheme we see that if $cr = 1$, then $\phi = kh$ and there is no dispersion. Otherwise, if $cr < 1$ we use the Taylor formula for the long waves $kh \ll 1$ and find the dispersion relation [cf. formula (10.58)]:

$$\begin{aligned}\phi &= \arctan\left[\frac{cr\sin(kh)}{1 - 2(cr)^2\sin^2(kh/2)}\right]\\ &= ck\tau\left[1 + (kh)^2\left(\frac{(cr)^2}{6} - \frac{1}{6}\right) + \mathcal{O}\left((kh)^4\right)\right].\end{aligned}$$

Hence, the phase speed for the Lax-Wendroff scheme is [cf. formula (10.59)]:

$$v_{\text{ph}} = -c\left[1 + (kh)^2\left(\frac{(cr)^2}{6} - \frac{1}{6}\right) + \mathcal{O}\left((kh)^4\right)\right].$$

We again conclude that the waves having different wavenumbers travel with different speeds, i.e., the propagation is accompanied by dispersion. Moreover, the long waves $kh \ll 1$ happen to travel slower than with the speed $-c$. The phase speed of these waves may only approach $-c$ as $kh \longrightarrow 0$.

10.2.4 Predictor-Corrector Schemes

When constructing difference schemes for time-dependent partial differential equations, one can exploit the same key idea that provides the foundation of Runge-Kutta schemes for ordinary differential equations. This is the idea of introducing intermediate stages of computation, or equivalently, of employing the predictor-corrector strategy, see Sections 9.2.6 and 9.4. This strategy allows one to increase the order of accuracy beyond the one that would have been obtained if the original scheme was used by itself, i.e., with no intermediate stages. Besides, in the case of quasi-linear differential equations this strategy facilitates design of the so-called conservative finite-difference schemes that will be discussed in Chapter 11.

Let us recall the idea of the predictor-corrector approach using one of the simplest Runge-Kutta schemes as an example; this scheme will be applied to solving the Cauchy problem for a first order ordinary differential equation:

$$\frac{dy}{dt} = G(t,y), \quad y(0) = \psi, \quad 0 \le t \le T. \tag{10.60}$$

If the value of the solution y_p at the moment of time $t_p = p\tau$ is already computed, then in order to compute y_{p+1} we first find the auxiliary quantity $\tilde{y}_{p+1/2}$ using the standard forward Euler scheme in the capacity of a predictor:

$$\frac{\tilde{y}_{p+1/2} - y_p}{\tau/2} = G(t_p, y_p). \tag{10.61}$$

Subsequently, we apply the corrector scheme to compute y_{p+1}:

$$\frac{y_{p+1} - y_p}{\tau} = G(t_{p+1/2}, \tilde{y}_{p+1/2}). \tag{10.62}$$

The auxiliary quantity $\tilde{y}_{p+1/2}$ obtained by scheme (10.61) with first order accuracy helps us approximately evaluate the slope of the integral curve at the midpoint of the interval $[t_p, t_{p+1}]$ and thus obtain y_{p+1} by formula (10.62) with accuracy higher than that of the Euler scheme (10.61).

We have already mentioned in Section 9.4.2 that all these considerations will remain valid if y_p, $\tilde{y}_{p+1/2}$, and y_{p+1} were to be interpreted as finite-dimensional vectors and G, accordingly, was to be thought of as a vector function. However, one can go even further and consider y_p, $\tilde{y}_{p+1/2}$, and y_{p+1} as elements of some functional space, and G as an operator acting in this space.

For instance, the Cauchy problem:

$$\begin{aligned} &\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0, \qquad -\infty < x < \infty, \quad 0 < t \le T, \\ &u(x,0) = \psi(x), \quad -\infty < x < \infty. \end{aligned} \tag{10.63}$$