According to (6.34), if $\tau > 0$ then the eigenvalues ν_j given by formula (6.35) are arranged in the descending order, see Figure 6.1:

$$1 > \nu_1 \geq \nu_2 \geq \ldots \geq \nu_n.$$

From Figure 6.1 it is also easy to see that the largest among the absolute values $|\nu_j|$, $j = 1, 2, \ldots, n$, may be either $|\nu_1| = |1 - \tau\lambda_1| \equiv |1 - \tau\lambda_{\min}|$ or $|\nu_n| = |1 - \tau\lambda_n| \equiv |1 - \tau\lambda_{\max}|$; the case $|\nu_n| = \max_j |\nu_j|$ is realized when $\nu_n = 1 - \tau\lambda_{\max} < 0$ and $|1 - \tau\lambda_{\max}| > |1 - \tau\lambda_{\min}|$. Consequently, the condition:

$$\rho = \max_j |\nu_j| < 1 \tag{6.36}$$

(see Lemma 6.1) coincides with the condition [see formula (6.32)]:

$$\rho = \max\{|1 - \tau\lambda_{\min}|, |1 - \tau\lambda_{\max}|\} < 1.$$

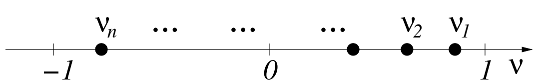

FIGURE 6.1: Eigenvalues of the matrix $\boldsymbol{B} = \boldsymbol{I} - \tau\boldsymbol{A}$.

Clearly, if $\tau > 0$ we can only guarantee $\rho < 1$ provided that the point ν_n on Figure 6.1 is located to the right of the point -1, i.e., if $\nu_n = 1 - \tau\lambda_{\max} > -1$. This means that along with $\tau > 0$ the second inequality of (6.30) also holds. Otherwise, if $\tau \geq 2/\lambda_{\max}$, then $\rho > 1$. If $\tau < 0$, then $\nu_j = 1 - \tau\lambda_j = 1 + |\tau|\lambda_j > 1$ for all $j = 1, 2, \ldots, n$, and we will always have $\rho = \max_j |\nu_j| > 1$. Hence, condition (6.30) is equivalent to the requirement (6.36) for $\boldsymbol{B} = \boldsymbol{I} - \tau\boldsymbol{A}$ (or to requirement (6.21) of Theorem 6.2). Therefore, by virtue of Lemma 6.1, have proven the first three implications of Theorem 6.3.

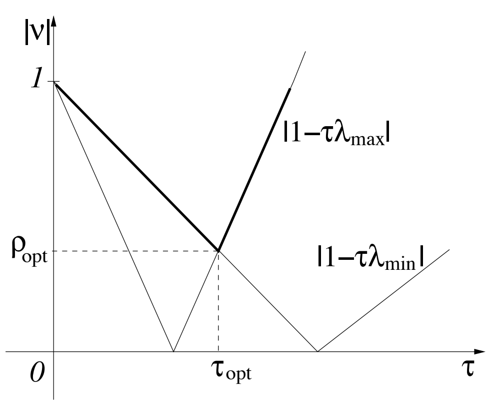

FIGURE 6.2: $|\nu_1|$ and $|\nu_n|$ as functions of τ.

To prove the remaining fourth implication, we need to analyze the behavior of the quantities $|\nu_1| = |1 - \tau\lambda_{\min}|$ and $|\nu_n| = |1 - \tau\lambda_{\max}|$ as functions of τ. We schematically show this behavior in Figure 6.2. From this figure, we determine that for smaller values of τ the quantity $|\nu_1|$ dominates, i.e., $|1 - \tau\lambda_{\min}| > |1 - \tau\lambda_{\max}|$, whereas for larger values of τ the quantity $|\nu_n|$ dominates, i.e., $|1 - \tau\lambda_{\max}| > |1 - \tau\lambda_{\min}|$. The value of $\rho(\tau) = \max\{|1 - \tau\lambda_{\min}|, |1 - \tau\lambda_{\max}|\}$ is shown by a bold polygonal line in Figure 6.2; it coincides with $|1-\tau\lambda_{\min}|$ before the intersection point, and after this point it coincides with $|1-\tau\lambda_{\max}|$. Consequently, the minimum value of $\rho = \rho_{\text{opt}}$ is achieved precisely at the intersection, i.e., at the value of $\tau = \tau_{\text{opt}}$ obtained from the following condition: $\nu_1(\tau) = |\nu_n(\tau)| = -\nu_n(\tau)$. This condition reads:

$$1 - \tau\lambda_{\min} = \tau\lambda_{\max} - 1,$$

which yields:

$$\tau_{\text{opt}} = \frac{2}{\lambda_{\min} + \lambda_{\max}}.$$

Consequently,

$$\rho_{\text{opt}} = \rho(\tau_{\text{opt}}) = 1 - \tau_{\text{opt}}\lambda_{\min} = \frac{\lambda_{\max} - \lambda_{\min}}{\lambda_{\max} + \lambda_{\min}} = \frac{\mu(\boldsymbol{A}) - 1}{\mu(\boldsymbol{A}) + 1}.$$

This expression is identical to (6.33), which completes the proof. ∎

Let us emphasize the following important consideration. Previously, we saw that the condition number of a matrix determines how sensitive the solution of the corresponding linear system will be to the perturbations of the input data (Section 5.3.2). The result of Theorem 6.3 provides the first evidence that the condition number also determines the rate of convergence of an iterative method. Indeed, from formula (6.33) it is clear that the closer the value of $\mu(\boldsymbol{A})$ to one, the closer the value of ρ_{opt} to zero, and consequently, the faster is the decay of the error according to estimate (6.31). When the condition number $\mu(\boldsymbol{A})$ increases, so does the quantity ρ_{opt} (while still remaining less than one) and the convergence slows down.

According to formulae (6.31) and (6.33), the optimal choice of the iteration parameter $\tau = \tau_{\text{opt}}$ enables the following error estimate:

$$\|\boldsymbol{\varepsilon}^{(p)}\| \le \left(\frac{1-\xi}{1+\xi}\right)^p \|\boldsymbol{\varepsilon}^{(0)}\|, \quad \text{where} \quad \xi = \frac{\lambda_{\min}}{\lambda_{\max}} = \frac{1}{\mu(\boldsymbol{A})}.$$

Moreover, Lemma 6.1 implies that this estimate cannot be improved, i.e., that there is a particular initial guess $\boldsymbol{x}^{(0)}$ (and hence $\boldsymbol{\varepsilon}^{(0)}$), for which the inequality transforms into a precise equality. Therefore, in order to guarantee that the initial error drops by a prescribed factor in the course of the iteration, i.e., in order to guarantee the estimate:

$$\|\boldsymbol{\varepsilon}^{(p)}\| \le \sigma \|\boldsymbol{\varepsilon}^{(0)}\|, \tag{6.37}$$

where $\sigma > 0$ is given, it is necessary and sufficient to select p that would satisfy:

$$\left(\frac{1-\xi}{1+\xi}\right)^p \le \sigma, \quad \text{i.e.,} \quad p \ge -\frac{\ln\sigma}{\ln(1+\xi) - \ln(1-\xi)}.$$

A more practical estimate for the number p can also be obtained. Note that

$$\ln(1+\xi) - \ln(1-\xi) = 2\xi \sum_{k=0}^{\infty} \frac{\xi^{2k}}{2k+1},$$