Let us write down equation number k from the system $\boldsymbol{Ax}=\boldsymbol{f}$:

$$a_{k1}x_1 + a_{k2}x_2 + \ldots + a_{kn}x_n = f_k.$$

Taking into account that $|x_k| \geq |x_j|$ for $j = 1, 2, \ldots, n$, we arrive at the following estimate:

$$\begin{aligned} |f_k| &= \left| \sum_j a_{kj}x_j \right| \geq |a_{kk}||x_k| - \sum_{j \neq k} |a_{kj}||x_j| \\ &\geq |a_{kk}||x_k| - \left(\sum_{j \neq k} |a_{kj}| \right) |x_k| = \left(|a_{kk}| - \sum_{j \neq k} |a_{kj}| \right) |x_k| \geq \delta |x_k|. \end{aligned}$$

Consequently, $|x_k| \leq |f_k|/\delta$. On the other hand, $|x_k| = \max_j |x_j| = \|\boldsymbol{x}\|_\infty$ and $|f_k| \leq \max_i |f_i| = \|\boldsymbol{f}\|_\infty$. Therefore,

$$\|\boldsymbol{x}\|_\infty \leq \frac{1}{\delta} \|\boldsymbol{f}\|_\infty. \tag{5.42}$$

In particular, estimate (5.42) means that if $\boldsymbol{f} = \boldsymbol{0} \in \mathbb{L}$ (e.g., $\mathbb{L} = \mathbb{R}^n$ of $\mathbb{L} = \mathbb{C}^n$), then $\|\boldsymbol{x}\|_\infty = 0$, and consequently, the homogeneous system $\boldsymbol{Ax} = \boldsymbol{0}$ only has a trivial solution $\boldsymbol{x} = \boldsymbol{0}$. As such, the inhomogeneous system $\boldsymbol{Ax} = \boldsymbol{f}$ has a unique solution for every $\boldsymbol{f} \in \mathbb{L}$. In other words, the inverse matrix $\boldsymbol{A}^{-1}$ exists.

Estimate (5.42) also implies that for any $\boldsymbol{f} \in \mathbb{L}, \boldsymbol{f} \neq \boldsymbol{0}$, the following estimate holds for $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{f}$:

$$\|\boldsymbol{A}^{-1}\boldsymbol{f}\|_\infty \leq \frac{1}{\delta} \|\boldsymbol{f}\|_\infty \quad \Longrightarrow \quad \frac{\|\boldsymbol{A}^{-1}\boldsymbol{f}\|_\infty}{\|\boldsymbol{f}\|_\infty} \leq \frac{1}{\delta},$$

so that

$$\|\boldsymbol{A}^{-1}\|_\infty = \max_{\boldsymbol{f} \in \mathbb{L}, \boldsymbol{f} \neq \boldsymbol{0}} \frac{\|\boldsymbol{A}^{-1}\boldsymbol{f}\|_\infty}{\|\boldsymbol{f}\|_\infty} \leq \frac{1}{\delta}.$$

$\square$

COROLLARY 5.1

Let $\boldsymbol{A}$ be a matrix with diagonal dominance of magnitude $\delta > 0$. Then,

$$\mu_\infty(\boldsymbol{A}) = \|\boldsymbol{A}\|_\infty \|\boldsymbol{A}^{-1}\|_\infty \leq \frac{1}{\delta} \|\boldsymbol{A}\|_\infty. \tag{5.43}$$

The proof is obtained as an immediate implication of the result of Theorem 5.5.

Exercises

1. Prove that the condition numbers $\mu_\infty(\boldsymbol{A})$ and $\mu_1(\boldsymbol{A})$ of the matrix $\boldsymbol{A}$ will not change after any permutation of rows and/or columns.

2. Prove that for a square matrix $\boldsymbol{A}$ and its transpose $\boldsymbol{A}^T$, the following equalities hold: $\mu_\infty(\boldsymbol{A}) = \mu_1(\boldsymbol{A}^T)$, $\mu_1(\boldsymbol{A}) = \mu_\infty(\boldsymbol{A}^T)$.
3. Show that the condition number of the operator $\boldsymbol{A}$ does not change if the operator is multiplied by an arbitrary non-zero real number.
4. Let $\mathbb{L}$ be a Euclidean space, and let $\boldsymbol{A} : \mathbb{L} \longmapsto \mathbb{L}$. Show that the condition number $\mu_{\boldsymbol{B}}(\boldsymbol{A}) = 1$ if and only if at least one of the following conditions holds:

   a) $\boldsymbol{A} = \alpha \boldsymbol{I}$, where $\alpha \in \mathbb{R}$;

   b) $\boldsymbol{A}$ is an orthogonal operator, i.e., $\forall \boldsymbol{x} \in \mathbb{L} : [\boldsymbol{Ax}, \boldsymbol{Ax}]_{\boldsymbol{B}} = [\boldsymbol{x}, \boldsymbol{x}]_{\boldsymbol{B}}$.

   c) $\boldsymbol{A}$ is a composition of $\alpha\boldsymbol{I}$ and an orthogonal operator.

5.* Prove that $\mu_{\boldsymbol{B}}(\boldsymbol{A}) = \mu_{\boldsymbol{B}}(\boldsymbol{A}^*_{\boldsymbol{B}})$, where $\boldsymbol{A}^*_{\boldsymbol{B}}$ is the operator adjoint to $\boldsymbol{A}$ in the sense of the scalar product $[\boldsymbol{x}, \boldsymbol{y}]_{\boldsymbol{B}}$.

6. Let $\boldsymbol{A}$ be a non-singular matrix, $\det \boldsymbol{A} \neq 0$. Multiply one row of the matrix $\boldsymbol{A}$ by some scalar α, and denote the new matrix by $\boldsymbol{A}_\alpha$. Show that $\mu(\boldsymbol{A}_\alpha) \longrightarrow \infty$ as $\alpha \longrightarrow \infty$.

7.* Prove that for any linear operator $\boldsymbol{A} : \mathbb{L} \longmapsto \mathbb{L}$:

$$\mu_{\boldsymbol{B}}(\boldsymbol{A}^*_{\boldsymbol{B}}\boldsymbol{A}) = (\mu_{\boldsymbol{B}}(\boldsymbol{A}))^2,$$

   where $\boldsymbol{A}^*_{\boldsymbol{B}}$ is the operator adjoint to $\boldsymbol{A}$ in the sense of the scalar product $[\boldsymbol{x}, \boldsymbol{y}]_{\boldsymbol{B}}$.

8.* Let $\boldsymbol{A} = \boldsymbol{A}^* > 0$ and $\boldsymbol{B} = \boldsymbol{B}^* > 0$ in the sense of some scalar product introduced on the linear space $\mathbb{L}$. Let the following inequalities hold for every $\boldsymbol{x} \in \mathbb{L}$:

$$\gamma_1(\boldsymbol{Bx}, \boldsymbol{x}) \leq (\boldsymbol{Ax}, \boldsymbol{x}) \leq \gamma_2(\boldsymbol{Bx}, \boldsymbol{x}),$$

   where $\gamma_1 > 0$ and $\gamma_2 > 0$ are two real numbers. Consider the operator $\boldsymbol{C} = \boldsymbol{B}^{-1}\boldsymbol{A}$ and prove that the condition number $\mu_{\boldsymbol{B}}(\boldsymbol{C})$ satisfies the estimate:

$$\mu_{\boldsymbol{B}}(\boldsymbol{C}) \leq \frac{\gamma_2}{\gamma_1}.$$

   Remark. We will solve this problem in Section 6.1.4 as it has numerous applications.

5.4 Gaussian Elimination and Its Tri-Diagonal Version

We will describe both the standard Gaussian elimination algorithm and the Gaussian elimination with pivoting, as they apply to solving an $n \times n$ system of linear algebraic equations in its canonical form:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= f_1, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= f_n. \end{aligned} \tag{5.44}$$

Recall that the Gaussian elimination procedures belong to the class of direct methods, i.e., they produce the exact solution of system (5.44) after a finite number of arithmetic operations.