onto $[-1,1]$ by a simple linear transformation: $x = \frac{a+b}{2} + t\frac{b-a}{2}$, where $-1 \le t \le 1$. Therefore, analyzing both interpolation and quadratures only on the interval $[-1,1]$ (as opposed to $[a,b]$) does not imply any loss of generality.

The $n+1$ roots of the Chebyshev polynomial $T_{n+1}(x)$ on the interval $[-1,1]$ are given by the formula:

$$x_m = \cos\frac{\pi(2m+1)}{2(n+1)}, \quad m = 0,1,2,\ldots,n, \tag{4.17}$$

and are referred to as the Chebyshev-Gauss nodes. The $n+1$ extrema of the Chebyshev polynomial $T_n(x)$ are given by the formula:

$$x_m = \cos\frac{\pi}{n}m, \quad m = 0,1,2,\ldots,n, \tag{4.18}$$

and are referred to as the Chebyshev-Gauss-Lobatto nodes. To employ Chebyshev polynomials in the context of numerical quadratures, we will additionally need their orthogonality property given by Lemma 4.1.

LEMMA 4.1
Chebyshev polynomials are orthogonal on the interval $-1 \le x \le 1$ with the weight $w(x) = 1/\sqrt{1-x^2}$, i.e., the following equalities hold for $k, l = 0,1,2,\ldots$:

$$\int_{-1}^{1} \frac{T_k(x)T_l(x)}{\sqrt{1-x^2}}dx = \begin{cases} \pi, & k = l = 0, \\ \pi/2, & k = l \ne 0, \\ 0, & k \ne l. \end{cases} \tag{4.19}$$

PROOF Let $x = \cos\varphi \;\Leftrightarrow\; \varphi = \arccos x$. Changing the variable in the integral, we obtain:

$$\begin{aligned} \int_{-1}^{1} \frac{T_k(x)T_l(x)}{\sqrt{1-x^2}}dx &= \int_{\pi}^{0} \frac{\cos(k\varphi)\cos(l\varphi)}{|\sin\varphi|}(-\sin\varphi)d\varphi = \int_{0}^{\pi} \cos(k\varphi)\cos(l\varphi)d\varphi \\ &= \frac{1}{2}\int_{0}^{\pi} [\cos(k+l)\varphi + \cos(k-l)\varphi]d\varphi = \begin{cases} \pi, & k = l = 0, \\ \pi/2, & k = l \ne 0, \\ 0, & k \ne l, \end{cases} \end{aligned}$$

because Chebyshev polynomials are only considered for $k \ge 0$ and $l \ge 0$. $\square$

Let the function $f = f(x)$ be defined for $-1 \le x \le 1$. We will approximate its definite integral over $[-1,1]$ taken with the weight $w(x) = 1/\sqrt{1-x^2}$ by integrating the Chebyshev interpolating polynomial $P_n(x,f)$ built on the Gauss nodes (4.17):

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}dx \approx \int_{-1}^{1} \frac{P_n(x,f)}{\sqrt{1-x^2}}dx. \tag{4.20}$$

The key point, of course, is to obtain a convenient expression for the integral on the right-hand side of formula (4.20) via the function values $f(x_m)$, $m = 0, 1, 2, \dots, m$, sampled on the Gauss grid (4.17). It is precisely the introduction of the weight $w(x) = 1/\sqrt{1-x^2}$ into the integral (4.20) that enables a particularly straightforward integration of the interpolating polynomial $P_n(x, f)$ over $[-1, 1]$.

LEMMA 4.2
Let the function $f = f(x)$ be defined on $[-1,1]$, and let $P_n(x, f)$ be its algebraic interpolating polynomial on the Chebyshev-Gauss grid (4.17). Then,

$$\int_{-1}^{1} \frac{P_n(x,f)}{\sqrt{1-x^2}} dx = \frac{\pi}{n+1} \sum_{m=0}^{n} f(x_m). \tag{4.21}$$

PROOF Recall that $P_n(x, f)$ is an interpolating polynomial of degree no higher than n built for the function $f(x)$ on the Gauss grid (4.17). The grid has a total of $n+1$ nodes, and the polynomial is unique. As shown in Section 3.2.3, see formulae (3.62) and (3.63) on page 76, $P_n(x, f)$ can be represented as:

$$P_n(x,f) = \sum_{k=0}^{n} a_k T_k(x),$$

where $T_k(x)$ are Chebyshev polynomials of degree k, and the coefficients a_k are given by:

$$a_0 = \frac{1}{n+1} \sum_{m=0}^{n} f_m T_0(x_m) \quad \text{and} \quad a_k = \frac{2}{n+1} \sum_{m=0}^{n} f_m T_k(x_m), \; k = 1, \dots, n.$$

Accordingly, it will be sufficient to show that equality (4.21) holds for all individual $T_k(x)$, $k = 0, 1, 2, \dots, n$:

$$\int_{-1}^{1} \frac{T_k(x)}{\sqrt{1-x^2}} dx = \frac{\pi}{n+1} \sum_{m=0}^{n} T_k(x_m). \tag{4.22}$$

Let $k = 0$, then $T_0(x) \equiv 1$. In this case Lemma 4.1 implies that the left-hand side of (4.22) is equal to π, and the right-hand side also appears equal to π by direct substitution. For $k > 0$, orthogonality (4.19) means that the left-hand side of (4.22) is equal to zero. To prove that the right-hand side is zero as well, we employ formula (3.22). The range of summation in (3.22) is from $m = 0$ to $m = N-1$, where $N = 2(n+1)$. As, however, cosine is an even function, the same result will hold for half the summation range and any $k = 1, 2, \dots, 2n+1$:

$$\begin{aligned}\sum_{m=0}^{n} T_k(x_m) &= \sum_{m=0}^{n} \cos\left(k \arccos\left[\cos\frac{\pi(2m+1)}{2(n+1)}\right]\right) \\ &= \sum_{m=0}^{n} \cos\left(k\frac{\pi(2m+1)}{2(n+1)}\right) = \sum_{m=0}^{n} \cos\left(k\frac{2\pi m}{2(n+1)} + k\frac{\pi}{2(n+1)}\right) = 0.\end{aligned}$$

This implies, in particular, that the right-hand side of (4.22) is zero for all $k = 1, 2, \dots, n$. Thus, we have established equality (4.21). ∎

Lemma 4.21 allows us to recast the approximate expression (4.20) for the integral $\int_{-1}^{1} f(x)w(x)dx$ as follows:

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}dx \approx \frac{\pi}{n+1}\sum_{m=0}^{n} f(x_m). \tag{4.23}$$

Formula (4.23) is known as the Gaussian quadrature formula with the weight $w(x) = 1/\sqrt{1-x^2}$ on the Chebyshev-Gauss grid (4.17). It has a particularly simple structure, which is very convenient for implementation. In practice, if we need to evaluate the integral $\int_{-1}^{1} f(x)dx$ for a given $f(x)$ with no weight, we introduce a new function $g(x) = f(x)\sqrt{1-x^2}$ and then rewrite formula (4.23) as:

$$\int_{-1}^{1} f(x)dx = \int_{-1}^{1} \frac{g(x)}{\sqrt{1-x^2}}dx \approx \frac{\pi}{n+1}\sum_{m=0}^{n} g(x_m).$$

A key advantage of the Gaussian quadrature (4.23) compared to the quadrature formulae studied previously in Section 4.1 is that the Gaussian quadrature does not get saturated by smoothness. Indeed, according to the following theorem (see also Remark 4.2 right after the proof of Theorem 4.5), the integration error automatically adjusts to the regularity of the integrand.

THEOREM 4.5

Let the function $f = f(x)$ be defined for $-1 \leq x \leq 1$; let it have continuous derivatives up to the order $r > 0$, and a square integrable derivative of order $r+1$:

$$\int_{-1}^{1} \left[f^{(r+1)}(x)\right]^2 dx < \infty.$$

Then, the error of the Gaussian quadrature (4.23) can be estimated as:

$$\left|\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}dx - \frac{\pi}{n+1}\sum_{m=0}^{n} f(x_m)\right| \leq \pi\frac{\zeta_n}{n^{r-1/2}}, \tag{4.24}$$

where $\zeta_n = o(1)$ as $n \longrightarrow \infty$.

PROOF The proof of inequality (4.24) is based on the error estimate (3.65) obtained in Section 3.2.4 (see page 77) for the Chebyshev algebraic interpolation. Namely, let $R_n(x) = f(x) - P_n(x, f)$. Then, under the assumptions of the current theorem we have:

$$\max_{-1\le x\le 1}|R_n(x)| \le \frac{\zeta_n}{n^{r-1/2}}, \quad \text{where} \quad \zeta_n = o(1), \quad n \longrightarrow \infty. \tag{4.25}$$

Consequently,

$$\left|\int_{-1}^{1}\frac{f(x)}{\sqrt{1-x^2}}dx - \frac{\pi}{n+1}\sum_{m=0}^{n}f(x_m)\right| = \left|\int_{-1}^{1}\frac{f(x)}{\sqrt{1-x^2}}dx - \int_{-1}^{1}\frac{P_n(x,f)}{\sqrt{1-x^2}}dx\right|$$
$$\le \int_{-1}^{1}\frac{|f(x)-P_n(x,f)|}{\sqrt{1-x^2}}dx \le \max_{-1\le x\le 1}|R_n(x)|\cdot\int_{-1}^{1}\frac{dx}{\sqrt{1-x^2}} \le \pi\frac{\zeta_n}{n^{r-1/2}},$$

which yields the desired estimate (4.24). ∎

Theorem 4.5 implies, in particular, that if the function $f = f(x)$ is infinitely differentiable on $[-1,1]$, then the Gaussian quadrature (4.23) exhibits a spectral rate of convergence as the dimension of the grid n increases. In other words, the integration error in (4.24) will decay faster than $\mathscr{O}(n^{-r+1/2})$ for any $r > 0$ as $n \longrightarrow \infty$.

REMARK 4.2 Error estimate (4.25) for the algebraic interpolation on Chebyshev grids can, in fact, be improved. According to formula (3.67), see Remark 3.2 on page 78, instead of inequality (4.25) we can write:

$$\max_{-1\le x\le 1}|R_n(x)| = o\left(\frac{\ln n}{n^{r+1/2}}\right) \quad \text{as} \quad n \longrightarrow \infty.$$

Then, the same argument as employed in the proof of Theorem 4.5 yields an improved convergence result for Gaussian quadratures:

$$\left|\int_{-1}^{1}\frac{f(x)}{\sqrt{1-x^2}}dx - \frac{\pi}{n+1}\sum_{m=0}^{n}f(x_m)\right| = o\left(\frac{\pi\ln n}{n^{r+1/2}}\right) \quad \text{as} \quad n \longrightarrow \infty,$$

where $r+1$ is the maximum number of derivatives that the integrand $f(x)$ has on the interval $-1 \le x \le 1$. However, even this is not the best estimate yet.

As shown in Section 3.2.7, by combining the Jackson inequality [Theorem 3.8, formula (3.79)], the Lebesgue inequality [Theorem 3.10, formula (3.80)], and estimate (3.83) for the Lebesgue constants given by the Bernstein theorem [Theorem 3.12], one can obtain the following error estimate for the Chebyshev algebraic interpolation:

$$\max_{-1\le x\le 1}|R_n(x)| = \mathscr{O}\left(\frac{\ln(n+1)}{n^{r+1}}\right), \quad \text{as} \quad n \longrightarrow \infty,$$