Let us first apply the trapezoidal quadrature formula to the first integral on the right-hand side of (4.10). This can be done individually for each term in the sum (4.8). First of all, for the constant component $k=0$ we immediately derive:

$$h\sum_{l=0}^{n-1}\left(\frac{\alpha_0/2}{2}+\frac{\alpha_0/2}{2}\right)=\frac{L}{2}\alpha_0=\int_0^L S_{n-1}(x)dx=\int_0^L f(x)dx.$$

For all other terms $k=1,2,\dots,n-1$, we exploit periodicity with the period L, use the definition of the grid $h=L/n$, and obtain:

$$h\sum_{l=0}^{n-1}\left(\frac{1}{2}\cos\frac{2\pi klh}{L}+\frac{1}{2}\cos\frac{2\pi k(l+1)h}{L}\right)=h\sum_{l=0}^{n-1}\cos\frac{2\pi klh}{L}$$
$$=\frac{h}{2}\sum_{l=0}^{n-1}\left(e^{i\frac{2\pi klh}{L}}+e^{-i\frac{2\pi klh}{L}}\right)=\frac{h}{2}\left(\frac{1-e^{i\frac{2\pi knh}{L}}}{1-e^{i\frac{2\pi kh}{L}}}+\frac{1-e^{-i\frac{2\pi knh}{L}}}{1-e^{-i\frac{2\pi kh}{L}}}\right)=0.$$

Analogously,

$$h\sum_{l=0}^{n-1}\left(\frac{1}{2}\sin\frac{2\pi klh}{L}+\frac{1}{2}\sin\frac{2\pi k(l+1)h}{L}\right)=0.$$

Altogether we conclude that the trapezoidal rule integrates the partial sum $S_{n-1}(x)$ given by formula (4.8) exactly:

$$h\sum_{l=0}^{n-1}\left(\frac{S_{n-1}(x_l)}{2}+\frac{S_{n-1}(x_{l+1})}{2}\right)=\int_0^L S_{n-1}(x)dx=\frac{L}{2}\alpha_0. \tag{4.11}$$

We also note that choosing the partial sum of order $n-1$, where the number of grid cells is n, is not accidental. From the previous derivation it is easy to see that equality (4.11) would no longer hold if we were to take $S_n(x)$ instead of $S_{n-1}(x)$.

Next, we need to apply the trapezoidal rule to the remainder of the series $\delta S_{n-1}(x)$ given by formula (4.9). Recall that the magnitude of this remainder, or equivalently, the rate of convergence of the Fourier series, is determined by the smoothness of the function $f(x)$. More precisely, as a part of the proof of Theorem 3.5 (page 68), we have shown that for the function $f(x)$ that has a square integrable derivative of order $r+1$, the following estimate holds, see formula (3.41):

$$\sup_{0\le x\le L}|\delta S_{n-1}(x)|\le\frac{\zeta_n}{n^{r+1/2}}, \tag{4.12}$$

where $\zeta_n=o(1)$ when $n\longrightarrow\infty$. Therefore,

$$\left|h\sum_{l=0}^{n-1}\left(\frac{\delta S_{n-1}(x_l)}{2}+\frac{\delta S_{n-1}(x_{l+1})}{2}\right)\right|\le\frac{h}{2}\sum_{l=0}^{n-1}\left(|\delta S_{n-1}(x_l)|+|\delta S_{n-1}(x_{l+1})|\right)$$
$$\le\frac{h}{2}2n\sup_{0\le x\le L}|\delta S_{n-1}(x)|\le L\cdot\frac{\zeta_n}{n^{r+1/2}}.$$