If we knew the quantity $y_0 = y(t_0)$ exactly, then we could have used the exact formula available for the concentration:

$$y(t) = y_0 e^{10(t-t_0)}. \tag{1.8}$$

We, however, only know the approximate value $y_0^* \approx y_0$ of the unknown quantity y_0. Therefore, instead of (1.8), the next best thing is to employ the approximate formula:

$$y^*(t) = y_0^* e^{10(t-t_0)}. \tag{1.9}$$

Clearly, the error $y^* - y$ of the approximate formula (1.9) is given by:

$$y^*(t) - y(t) = (y_0^* - y_0)e^{10(t-t_0)}, \quad 0 \leq t \leq 1.$$

Assume now that we need to measure y_0^* to the accuracy δ, $|y_0^* - y_0| < \delta$, that would be sufficient to guarantee an initially prescribed tolerance ε for determining $y(t)$ everywhere on the interval $0 \leq t \leq 1$, i.e., would guarantee the error estimate:

$$|y^*(t) - y(t)| < \varepsilon, \quad 0 \leq t \leq 1.$$

It is easy to see that $\max\limits_{0 \leq t \leq 1} |y^*(t) - y(t)| = |y^*(1) - y(1)| = |y_0^* - y_0| e^{10(1-t_0)}$. This yields the following constraint that the accuracy δ of measuring y_0 must satisfy:

$$\delta \leq \varepsilon e^{-10(1-t_0)}. \tag{1.10}$$

Let y_0 be measured at the moment of time $t_0 = 0$. Then, inequality (1.10) would imply that this measurement has to be e^{10} times, i.e., thousands of times, more accurate than the required guaranteed accuracy of the result ε. In other words, the answer $y(t)$ appears quite sensitive to the error in specifying the input data y_0, and the problem is ill conditioned.

On the other hand, if y_0 were to be measured at $t_0 = 1$, then $\delta = \varepsilon$, and it would be sufficient to conduct the measurement with a considerably lower accuracy than the one required in the case of $t_0 = 0$. This problem is well conditioned.

Exercises

1. Consider the problem of computing $y(x) = \frac{1+x}{1-x}$ as a function of x, for $x \in (1/2, 1)$ and also for $x \in (-1, 0)$. On which of the two intervals is this problem better conditioned with respect to the perturbations of x?
2. Let $y = \sqrt{2} - 1$. Equivalently, one can write $y = (\sqrt{2} + 1)^{-1}$. Which of the two formulae is more sensitive to the error when $\sqrt{2}$ is approximated by a finite decimal fraction? **Hint.** Compare absolute values of derivatives for the functions $(x - 1)$ and $(x + 1)^{-1}$.

1.3 Error

In any computational problem, one needs to find the solution given some appropriate input data. If the solution can be obtained with an ideal accuracy, then there is no