$$= \frac{1}{N}\sum_{m=0}^{N-1}\left[1+\cos\left(\frac{4\pi k}{N}m+\frac{2\pi k}{N}\right)\right]$$
$$= \frac{1}{N}\sum_{m=0}^{N-1}1+\frac{1}{N}\sum_{m=0}^{N-1}\cos 2k\left(\frac{2\pi}{N}m+\frac{\pi}{N}\right) = 1+0 = 1,$$

$$\begin{aligned}(\eta^{(k)},\eta^{(k)}) &= \frac{2}{N}\sum_{m=0}^{N-1}\sin^2\left(\frac{2\pi k}{N}m+\frac{\pi k}{N}\right)\\ &= \frac{1}{N}\sum_{m=0}^{N-1}\left[1-\cos\left(\frac{4\pi k}{N}m+\frac{2\pi k}{N}\right)\right] = 1.\end{aligned}$$

To prove equality (3.16), we first notice that for $r,s = 0,1,\dots,n$ and $r \neq s$ we always have $1 \le |r\pm s| \le N-1$, and then use formula (3.20) to obtain:

$$\begin{aligned}(\xi^{(r)},\xi^{(s)}) &= \frac{2}{N}\sum_{m=0}^{N-1}\cos\left\{r\left(\frac{2\pi m}{N}+\frac{\pi}{N}\right)\right\}\cdot\cos\left\{s\left(\frac{2\pi m}{N}+\frac{\pi}{N}\right)\right\}\\ &= \frac{1}{N}\sum_{m=0}^{N-1}\left[\cos\left\{(r+s)\left(\frac{2\pi m}{N}+\frac{\pi}{N}\right)\right\}+\cos\left\{(r-s)\left(\frac{2\pi m}{N}+\frac{\pi}{N}\right)\right\}\right] = 0.\end{aligned}$$

Equality (3.17) is proven similarly, except that instead of the trigonometric identity $2\cos\alpha\cos\beta = \cos(\alpha+\beta)+\cos(\alpha-\beta)$ that has been used when proving (3.16), one rather needs to employ another identity: $2\sin\alpha\sin\beta = \cos(\alpha+\beta)-\cos(\alpha-\beta)$. Finally, yet another trigonometric identity: $2\sin\alpha\cos\beta = \sin(\alpha+\beta)+\sin(\alpha-\beta)$ is to be used for proving formula (3.18).

Altogether, we have established that (3.13) is an orthonormal basis in the space F_N. Therefore, every function $f = \{f_m\} \in F_N$ can be represented as a linear combination of the basis functions (3.13):

$$f_m = \sum_{k=0}^{n} a_k\cos\frac{2\pi k}{L}x_m + \sum_{k=1}^{n+1} b_k\sin\frac{2\pi k}{L}x_m.$$

Calculating the dot products of both the left-hand side and the right-hand side of the previous equality with all the basis functions $\xi^{(r)}$ and $\eta^{(s)}$, $r = 0,1,\dots,n$, $s = 1,2,\dots,n+1$, we arrive at the equalities:

$$\begin{aligned} a_0 &= (f,\xi^{(0)}),\\ a_k &= \sqrt{2}(f,\xi^{(k)}), \qquad k = 1,2,\dots,n,\\ b_k &= \sqrt{2}(f,\eta^{(k)}), \qquad k = 1,2,\dots,n,\\ b_{n+1} &= (f,\eta^{(n+1)}),\end{aligned}$$

that, according to definitions (3.13), coincide with formulae (3.7)–(3.10). ∎