The coefficients of this polynomial are given by the formulae:

$$a_0 = \frac{1}{N}\sum_{m=0}^{N-1} f_m, \tag{3.7}$$

$$a_k = \frac{2}{N}\sum_{m=0}^{N-1} f_m \cos k\left(\frac{2\pi}{N}m + \frac{\pi}{N}\right), \qquad k = 1,2,\ldots,n, \tag{3.8}$$

$$b_k = \frac{2}{N}\sum_{m=0}^{N-1} f_m \sin k\left(\frac{2\pi}{N}m + \frac{\pi}{N}\right), \qquad k = 1,2,\ldots,n, \tag{3.9}$$

$$b_{n+1} = \frac{1}{N}\sum_{m=0}^{N-1} f_m(-1)^m. \tag{3.10}$$

PROOF Let us consider a set of all real valued periodic discrete functions:

$$f_{m+N} = f_m, \qquad m = 0, \pm 1, \pm 2, \ldots, \tag{3.11}$$

defined on the grid $x_m = \frac{L}{N}m + \frac{L}{2N}$. We will only be considering these functions on the grid interval $m = 0,1,\ldots,N-1$, because for all other m's they can be unambiguously reconstructed by virtue of periodicity (3.11).

The entire set of these functions, supplemented by the conventional operations of addition and multiplication by real scalars, form a linear space that we will denote F_N. The dimension of this space is equal to N, because the system of N linearly independent functions (vectors) $\tilde{\psi}^{(k)} \in F_N$, $k = 1,2,\ldots,N$:

$$\tilde{\psi}_m^{(k)} \stackrel{\text{def}}{=} \begin{cases} 0, & \text{if} \quad m \neq k-1, \\ 1, & \text{if} \quad m = k-1, \end{cases}$$

provides a basis in the space F_N. Indeed, any function $f \in F_N$, $f = \{f_m \,|\, m = 0,1,\ldots,N-1\}$ always admits a unique representation as a linear combination of the basis functions $\tilde{\psi}^{(k)}$: $f = \sum_{k=1}^{N} f_{k-1}\tilde{\psi}^{(k)}$.

Let us now introduce a Euclidean dot (i.e., inner) product in the space F_N:

$$(f,g) = \frac{1}{N}\sum_{m=0}^{N-1} f_m g_m, \tag{3.12}$$

and show that the system of $2(n+1)$ functions: $\xi^{(k)} = \{\xi_m^{(k)}\}$, $k = 0,1,\ldots,n$, and $\eta^{(k)} = \{\eta_m^{(k)}\}$, $k = 1,2,\ldots,n+1$, where

$$\begin{aligned} \xi_m^{(0)} &= \cos(0 \cdot x_m) \equiv 1, \\ \xi_m^{(k)} &= \sqrt{2}\cos\left(\frac{2\pi k}{L}x_m\right), \quad k = 1,2,\ldots,n, \\ \eta_m^{(k)} &= \sqrt{2}\sin\left(\frac{2\pi k}{L}x_m\right), \quad k = 1,2,\ldots,n, \\ \eta_m^{(n+1)} &= \sin\left(\frac{2\pi(n+1)}{L}x_m\right) = (-1)^m, \end{aligned} \tag{3.13}$$