Chapter 3

Trigonometric Interpolation

Along with the algebraic interpolation described in Chapter 2, one also uses interpolation by means of trigonometric polynomials of the type:

$$Q\left(\cos\frac{2\pi}{L}x, \sin\frac{2\pi}{L}x\right) = \sum_{k=0}^{n} a_k \cos\frac{2\pi k}{L}x + \sum_{k=1}^{n+1} b_k \sin\frac{2\pi k}{L}x, \tag{3.1}$$

where n is a positive integer, $L > 0$, and a_k & b_k are the coefficients. A trigonometric interpolating polynomial $Q\left(\cos\frac{2\pi}{L}x, \sin\frac{2\pi}{L}x\right)$ that would coincide with the given L-periodic function $f(x)$, $f(x+L) = f(x)$, at the equidistant interpolation nodes:

$$x_m = \frac{L}{N}m + x_0, \qquad m = 0, 1, \ldots, N-1, \qquad x_0 = \text{const}, \tag{3.2}$$

can be chosen such that it will have some important advantages compared to the algebraic interpolating polynomial built on the same grid (3.2).

First, the error of the trigonometric interpolation

$$R_N(x, f) \stackrel{\text{def}}{=} f(x) - Q\left(\cos\frac{2\pi}{L}x, \sin\frac{2\pi}{L}x\right) \tag{3.3}$$

converges to zero uniformly with respect to x as $N \longrightarrow \infty$ already if the second derivative of $f(x)$ is piecewise continuous.[1] Moreover, the rate of this convergence, i.e., the rate of decay of the error (3.3) as $N \longrightarrow \infty$, automatically takes into account the smoothness of $f(x)$, i.e., increases for those functions $f(x)$ that have more derivatives. Specifically, we will prove that

$$\max_x |R_N(x, f)| = \mathscr{O}\left(\frac{M_{r+1}}{N^{r-1/2}}\right), \quad \text{where} \quad M_{r+1} = \max_x \left|\frac{d^{r+1} f(x)}{dx^{r+1}}\right|.$$

Second, it turns out that the sensitivity of the trigonometric interpolating polynomial (3.1) to the errors committed when specifying the function values $f_m = f(x_m)$ on the grid (3.2) remains "practically flat" (i.e., grows slowly) as N increases.

The foregoing two properties — automatic improvement of accuracy for smoother functions, and slow growth of the Lebesgue constants that translates into numerical stability — are distinctly different from the properties of algebraic interpolation on

[1] In fact, even less regularity may be required of $f(x)$, see Section 3.2.7.