1.2 Conditioning

Speaking in most general terms, for any given problem one can basically identify the input data and the output result(s), i.e., the solution, so that the former determine the latter. In this book, we will mostly analyze problems for which the solution exists and is unique. If, in addition, the solution depends continuously on the data, i.e., if for a vanishing perturbation of the data the corresponding perturbation of the solution will also be vanishing, then the problem is said to be *well-posed.*

A somewhat more subtle characterization of the problem, on top of its well-posedness, is known as the *conditioning*. It has to do with quantifying the sensitivity of the solution, or some of its key characteristics, to perturbations of the input data. This sensitivity may vary strongly for different problems that could otherwise look very similar. If it is "low" (weak), then the problem is said to be *well conditioned;* if, conversely, the sensitivity is "high" then the problem is *ill conditioned.* The notions of low and high are, of course, problem-specific. We emphasize that the concept of conditioning pertains to both continuous and discrete problems. Typically, not only do ill conditioned problems require excessively accurate definition of the input data, but also appear more difficult for computations.

Consider, for example, the quadratic equation $x^2 - 2\alpha x + 1 = 0$ for $|\alpha| > 1$. It has two real roots that can be expressed as functions of the argument α: $x_{1,2} = \alpha \pm \sqrt{\alpha^2 - 1}$. We will interpret α as the datum in the problem, and $x_1 = x_1(\alpha)$ and $x_2 = x_2(\alpha)$ as the corresponding solution. Clearly, the sensitivity of the solution to the perturbations of α can be characterized by the magnitude of the derivatives $\frac{dx_{1,2}}{d\alpha} = 1 \pm \frac{\alpha}{\sqrt{\alpha^2-1}}$. Indeed, $\Delta x_{1,2} \approx \frac{dx_{1,2}}{d\alpha}\Delta\alpha$. We can easily see that the derivatives $\frac{dx_{1,2}}{d\alpha}$ are small for large $|\alpha|$, but they become large when $|\alpha|$ approaches 1. We can therefore conclude that the problem of finding the roots of $x^2 - 2\alpha x + 1 = 0$ is well conditioned when $|\alpha| \gg 1$, and ill conditioned when $|\alpha| = \mathscr{O}(1)$. We should also note that conditioning can be improved if, instead of the original quadratic equation, we consider its equivalent $x^2 - \frac{1+\beta^2}{\beta}x + 1 = 0$, where $\beta = \alpha + \sqrt{\alpha^2 - 1}$. In this case, $x_1 = \beta$ and $x_2 = \beta^{-1}$; the two roots coincide for $|\beta| = 1$, or equivalently, $|\alpha| = 1$. However, the problem of evaluating $\beta = \beta(\alpha)$ is still ill conditioned near $|\alpha| = 1$.

Our next example involves a simple ordinary differential equation. Let $y = y(t)$ be the concentration of some substance at the time t, and assume that it satisfies:

$$\frac{dy}{dt} - 10y = 0.$$

Let us take an arbitrary t_0, $0 \le t_0 \le 1$, and perform an approximate measurement of the actual concentration $y_0 = y(t_0)$ at this moment of time, thus obtaining:

$$y\big|_{t=t_0} = y_0^*.$$

Our overall task will be to determine the concentration $y = y(t)$ at all other moments of time t from the interval $[0, 1]$.