In particular, provided that the third derivative $f^{(3)}(x)$ is bounded on $[a,b]$, the cubic spline $\psi(x,2) \equiv \psi(x)$, along with its derivatives of orders $m = 1, 2$, will converge to the function $f(x)$ and its respective derivatives with the rate of $\mathscr{O}(h^{3-m})$.

On the other hand, the undesirable property of saturation by smoothness, which is inherent for both the classical piecewise polynomial interpolation and local splines, is shared by the Schoenberg splines as well, notwithstanding the loss of their local nature. Namely, the nonlocal splines of smoothness s on a uniform interpolation grid with size h will guarantee the order of error $\mathscr{O}(h^{s+1})$ for the functions that have a maximum of $s+1$ bounded derivatives, and they will not provide accuracy higher than $\mathscr{O}(h^{s+1})$ even for the functions $f(x)$ that have more than $s+1$ derivatives.

Even though the coefficients of a nonlocal spline on a given interval $[x_k, x_{k+1}]$ depend on the function values on the entire grid, it is known that in practice the influence of the remote nodes is rather weak. Nonetheless, to actually evaluate the coefficients, one needs to solve the full system (2.62). Therefore, a natural task of improving the accuracy of a Schoenberg's spline by adding a few interpolation nodes in a particular local region basically implies starting from the very beginning, i.e., writing down and then solving a new system of type (2.62). In contradistinction to that, the splines of Section 2.3.1 are particularly well suited for such local grid refinements. In doing so, the additional computational effort is merely proportional to the number of new nodes.

For further detail on the subject of splines we refer the reader to [dB01].

2.3.3 Proof of Theorem 2.11

This section can be skipped during the first reading.

The coefficients of the polynomial

$$Q_{2s+1}(x,k) = c_{0,k} + c_{1,k}x + \ldots + c_{2s+1,k}x^{2s+1} \tag{2.68}$$

are determined by solving the linear algebraic system (2.43), (2.44). The right-hand sides of the equations that compose sub-system (2.43) have the form

$$a_0^{(m)} f_{k-j} + a_1^{(m)} f_{k-j+1} + \ldots + a_s^{(m)} f_{k-j+s}, \qquad m = 0, 1, \ldots, s,$$

while those that pertain to sub-system (2.44) have the form

$$b_0^{(m)} f_{k-j+1} + b_1^{(m)} f_{k-j+2} + \ldots + b_s^{(m)} f_{k-j+s+1}, \qquad m = 0, 1, \ldots, s,$$

where $a_i^{(m)}$ and $b_i^{(m)}$, $i = 0, 1, \ldots, s$, $m = 0, 1, \ldots, s$, are some numbers that do not depend on[5] f_{k-j}, f_{k-j+1}, ..., $f_{k-j+s+1}$.

Consequently, one can say that the given quantities f_{k-j}, f_{k-j+1}, ..., $f_{k-j+s+1}$ determine the solution $c_{0,k}$, $c_{1,k}$, ..., $c_{2s+1,k}$ of system (2.43), (2.44) through the formulae of type

$$c_{r,k} = \alpha_0^{(r)} f_{k-j} + \alpha_1^{(r)} f_{k-j+1} + \ldots + \alpha_{s+1}^{(r)} f_{k-j+s+1}, \quad r = 0, 1, \ldots, 2s+1, \tag{2.69}$$

[5]Hereafter in this section, we will use the notation $f_l = f(x_l)$ for all nodes l.