true. We thus see that the difficulties in reducing boundary value problems for the Helmholtz equation to integral equations are again related to interior resonances, as in the case of the Laplace equation. These difficulties may translate into serious hurdles for computations.

Integral equations of the classical potential theory have been explicitly constructed for the boundary value problems of elasticity, see, e.g., [MMP95], for the Stokes system of equations (low speed flows of viscous fluid), see [Lad69], and for some other equations and systems, for which an analytical representation of the fundamental solution is available.

Along with exploiting various types of surface potentials, one can build boundary integral equations using the relation between the values of the solution $u|_\Gamma$ and its normal derivative $\frac{\partial u}{\partial n}\big|_\Gamma$ at the boundary Γ of the region Ω. This relation can be obtained with the help of the Green's formula:

$$u(\boldsymbol{x}) = \int_\Gamma \left(G(\boldsymbol{x}-\boldsymbol{y}) \frac{\partial u(\boldsymbol{y})}{\partial n_{\boldsymbol{y}}} - \frac{\partial G(\boldsymbol{x}-\boldsymbol{y})}{\partial n_{\boldsymbol{y}}} u(\boldsymbol{y}) \right) ds_{\boldsymbol{y}} \tag{13.8}$$

that represents the solution $u(\boldsymbol{x})$ of the Laplace or Helmholtz equation at some interior point $\boldsymbol{x} \in \Omega$ via the boundary values $u|_\Gamma$ and $\frac{\partial u}{\partial n}\big|_\Gamma$. One gets the desired integral equation by passing to the limit $\boldsymbol{x} \longrightarrow \boldsymbol{x}_0 \in \Gamma$ while taking into account the discontinuity of the double-layer potential, i.e., the jump of the second term on the right-hand side of (13.8), at the interface Γ. Then, in the case of a Neumann problem, one can substitute a known function $\frac{\partial u}{\partial n}\big|_\Gamma = \varphi(\boldsymbol{x})$ into the integral and arrive at a Fredholm integral equation of the second kind that can be solved with respect to $u|_\Gamma$. The advantage of doing so compared to solving equation (13.6b) is that the quantity to be computed is $u|_\Gamma$ rather than the auxiliary density $\rho(\boldsymbol{y})$.

13.2 Discretization of Integral Equations and Boundary Elements

Discretization of the boundary integral equations derived in Section 13.1 encounters difficulties caused by the singular behavior of the kernels $\frac{\partial G}{\partial n_{\boldsymbol{y}}}$ and G. In this section, we will use equation (13.6a) as an example and illustrate the construction of special quadrature formulae that approximate the corresponding integrals.

First, we need to triangulate the surface $\Gamma = \partial\Omega$, i.e., partition it into a finite number of non-overlapping curvilinear triangles. These triangles are called the boundary elements, and we denote them Γ_j, $j = 1, 2, \ldots, J$. They may only intersect by their sides or vertexes. Inside every boundary element the unknown density $\sigma(\boldsymbol{y})$ is assumed constant, and we denote its value σ_j, $j = 1, 2, \ldots, J$.

Next, we approximately replace the convolution integral in equation (13.6a) by