Solutions of the Dirichlet problems (13.1a) and (13.1c) are to be sought in the form of a double-layer potential (13.5), whereas solutions to the Neumann problems (13.1b) and (13.1d) are to be sought in the form of a single-layer potential (13.4). Then, the the so-called Fredholm integral equations of the second kind can be obtained for the unknown densities of the potentials (see, e.g., [TS63]). These equations read as follows: For the interior Dirichlet problem (13.1a):

$$\sigma(\boldsymbol{x}) - \frac{1}{2\pi}\int\limits_{\Gamma} \sigma(\boldsymbol{y})\frac{\partial}{\partial n_{\boldsymbol{y}}}\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}ds_{\boldsymbol{y}} = -\frac{1}{2\pi}\varphi(\boldsymbol{x}), \quad \boldsymbol{x}\in\Gamma, \tag{13.6a}$$

for the interior Neumann problem (13.1b):

$$\rho(\boldsymbol{x}) - \frac{1}{2\pi}\int\limits_{\Gamma} \rho(\boldsymbol{y})\frac{\partial}{\partial n_{\boldsymbol{y}}}\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}ds_{\boldsymbol{y}} = -\frac{1}{2\pi}\varphi(\boldsymbol{x}), \quad \boldsymbol{x}\in\Gamma, \tag{13.6b}$$

for the exterior Dirichlet problem (13.1c):

$$\sigma(\boldsymbol{x}) + \frac{1}{2\pi}\int\limits_{\Gamma} \sigma(\boldsymbol{y})\frac{\partial}{\partial n_{\boldsymbol{y}}}\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}ds_{\boldsymbol{y}} = \frac{1}{2\pi}\varphi(\boldsymbol{x}), \quad \boldsymbol{x}\in\Gamma, \tag{13.6c}$$

and for the exterior Neumann problem (13.1d):

$$\rho(\boldsymbol{x}) + \frac{1}{2\pi}\int\limits_{\Gamma} \rho(\boldsymbol{y})\frac{\partial}{\partial n_{\boldsymbol{y}}}\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}ds_{\boldsymbol{y}} = \frac{1}{2\pi}\varphi(\boldsymbol{x}), \quad \boldsymbol{x}\in\Gamma. \tag{13.6d}$$

In the framework of the classical potential theory, integral equations (13.6a) and (13.6d) are shown to be uniquely solvable; their respective solutions $\sigma(\boldsymbol{x})$ and $\rho(\boldsymbol{x})$ exist for any given function $\varphi(\boldsymbol{x})$, $\boldsymbol{x}\in\Gamma$. The situation is different for equations (13.6b) and (13.6c). Equation (13.6b) is only solvable if equality (13.3) holds. The latter constraint reflects on the nature of the interior Neumann problem (13.1b), which also has a solution only if the additional condition (13.3) is satisfied. As for the integral equation (13.6c), is not solvable for an arbitrary $\varphi(\boldsymbol{x})$ either, even though the exterior Dirichlet problem (13.1c) always has a unique solution. As such, transition from the boundary value problem (13.1c) to the integral equation (13.6c) may not always be justified.

The loss of solvability for the integral equation (13.6c) can be explained. When we look for a solution to the exterior Dirichlet problem (13.1c) in the form of a double-layer potential (13.5), we essentially require that the solution $u(\boldsymbol{x})$ decay at infinity as $\mathscr{O}(|\boldsymbol{x}|^{-2})$, because it is known that $W(\boldsymbol{x}) = \mathscr{O}(|\boldsymbol{x}|^{-2})$ as $|\boldsymbol{x}| \longrightarrow \infty$. However, the original formulation (13.1c), (13.2) only requires that the solution vanish at infinity. In other words, there may be solutions that satisfy (13.1c), (13.2), yet they decay slower than $\mathscr{O}(|\boldsymbol{x}|^{-2})$ when $|\boldsymbol{x}| \longrightarrow \infty$ and as such, are not captured by the integral equation (13.6c). This causes equation (13.6c) to lose solvability for some $\varphi(\boldsymbol{x})$.