Chapter 13

Boundary Integral Equations and the Method of Boundary Elements

In this chapter, we provide a very brief account of the classical potential theory and show how it can help reduce a given boundary value problem to an equivalent integral equation at the boundary of the original domain. We also address the issue of discretization for the corresponding integral equations, and identify the difficulties that limit the class of problems solvable by the method of boundary elements.

13.1 Reduction of Boundary Value Problems to Integral Equations

To illustrate the key concepts, it will be sufficient to consider the interior and exterior Dirichlet and Neumann boundary value problems for the Laplace equation:

$$\Delta u \equiv \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} = 0.$$

Let Ω be a bounded domain of the three-dimensional space $\mathbb{R}^3$, and assume that its boundary $\Gamma = \partial\Omega$ is sufficiently smooth. Let also Ω_1 be the complementary domain: $\Omega_1 = \mathbb{R}^3 \backslash \bar{\Omega}$. Consider the following four problems:

$$\Delta u = 0, \quad \boldsymbol{x} \in \Omega, \qquad u\big|_\Gamma = \varphi(\boldsymbol{x})\big|_{\boldsymbol{x}\in\Gamma}, \tag{13.1a}$$

$$\Delta u = 0, \quad \boldsymbol{x} \in \Omega, \qquad \frac{\partial u}{\partial n}\Big|_\Gamma = \varphi(\boldsymbol{x})\big|_{\boldsymbol{x}\in\Gamma}, \tag{13.1b}$$

$$\Delta u = 0, \quad \boldsymbol{x} \in \Omega_1, \qquad u\big|_\Gamma = \varphi(\boldsymbol{x})\big|_{\boldsymbol{x}\in\Gamma}, \tag{13.1c}$$

$$\Delta u = 0, \quad \boldsymbol{x} \in \Omega_1, \qquad \frac{\partial u}{\partial n}\Big|_\Gamma = \varphi(\boldsymbol{x})\big|_{\boldsymbol{x}\in\Gamma}, \tag{13.1d}$$

where $\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$, n is the outward normal to Γ, and $\varphi(\boldsymbol{x})$ is a given function for $\boldsymbol{x} \in \Gamma$. Problems (13.1a) and (13.1c) are the interior and exterior Dirichlet problems, respectively, and problems (13.1b) and (13.1d) are the interior and exterior Neumann problems, respectively. For the exterior problems (13.1c) and (13.1d), we also need to specify the desired behavior of the solution at infinity:

$$u(\boldsymbol{x}) \longrightarrow 0, \;\text{ as }\; |\boldsymbol{x}| \equiv (x_1^2 + x_2^2 + x_3^2)^{1/2} \longrightarrow \infty. \tag{13.2}$$