approximately replaced by difference quotients according to the formulae:

$$\begin{aligned}
&\frac{\partial}{\partial x}\left(a(x,y)\frac{\partial u(x,y)}{\partial x}\right) \approx \tilde{\mathbf{\Lambda}}_{xx}u(x,y) \\
&\stackrel{\text{def}}{=} \frac{1}{h}\left[a\Big(x+\frac{h}{2},y\Big)\frac{u(x+h,y)-u(x,y)}{h} - a\Big(x-\frac{h}{2},y\Big)\frac{u(x,y)-u(x-h,y)}{h}\right], \\
&\frac{\partial}{\partial y}\left(b(x,y)\frac{\partial u(x,y)}{\partial y}\right) \approx \tilde{\mathbf{\Lambda}}_{yy}u(x,y) \\
&\stackrel{\text{def}}{=} \frac{1}{h}\left[b\Big(x,y+\frac{h}{2}\Big)\frac{u(x,y+h)-u(x,y)}{h} - b\Big(x,y-\frac{h}{2}\Big)\frac{u(x,y)-u(x,y-h)}{h}\right].
\end{aligned}$$

Then, we obtain a scheme of type (12.5):

$$\boldsymbol{L}_h u^{(h)} = f^{(h)},$$

where

$$\begin{aligned}
\boldsymbol{L}_h u^{(h)} &= \begin{cases} \tilde{\mathbf{\Lambda}}_{xx}u^{(h)}\Big|_{(x_m,y_n)} + \tilde{\mathbf{\Lambda}}_{yy}u^{(h)}\Big|_{(x_m,y_n)}, & (x_m,y_n)\in\Omega_h^0, \\ u_{mn}, & (x_m,y_n)\in\Gamma_h, \end{cases} \\
f^{(h)} &= \begin{cases} \varphi(x_m,y_n), & (x_m,y_n)\in\Omega_h^0, \\ \psi(s_{mn}), & (x_m,y_n)\in\Gamma_h. \end{cases}
\end{aligned} \tag{12.13}$$

Using the Taylor formula, one can make sure that this scheme is second order accurate. Stability of this scheme can also be proven similarly to how it was done Section 12.1.2. In doing so, some additional issues will have to be addressed.

In practice, when solving real application-driven problems on the computer, one normally conducts theoretical analysis only for simple model formulations of the type outlined above. As for the actual error estimates, they are most often obtained experimentally, by comparing the results of computations on the grids with different values of the size h.

Exercises

1. Let the function $v^{(h)} = \{v_{mn}\}$ be defined on the grid Ω_h, and let it satisfy the finite-difference Laplace equation:

   $$\mathbf{\Lambda}_h v^{(h)}\big|_{(x_m,y_n)} = 0, \quad (x_m,y_n)\in\Omega_h^0,$$

   at the interior nodes $(x_m,y_n) = (mh,nh)$ of the grid. Prove that either $v^{(h)} = \text{const}$ everywhere on Ω_h, or the maximum and minimum values of $v^{(h)}$ are not attained at any interior grid node (a strengthened maximum principle).

2. Prove that if the inequality $\mathbf{\Lambda}_h v^{(h)} \geq 0$ holds for all interior nodes of the grid Ω_h, and if this inequality becomes strict at least at one node from Ω_h^0, then $v^{(h)}$ does not attain its maximum value at any interior node.