$v_{xx}+v_{yy}=0$. It is known that the solution $v(x,y)$ assumes its maximum and minimum values at the boundary of the region where this solution is defined.

The maximum principle implies that the problem:

$$\boldsymbol{L}_h u^{(h)} = 0,$$

where

$$\boldsymbol{L}_h u^{(h)} = \begin{cases} \boldsymbol{\Lambda}_h u^{(h)}\big|_{(x_m,y_n)}, & (x_m,y_n)\in\Omega_h^0, \\ u^{(h)}\big|_{(x_m,y_n)}, & (x_m,y_n)\in\Gamma_h, \end{cases}$$

may only have a trivial solution, $u^{(h)}\equiv 0$, because the maximum and minimum values of this solution are attained at the boundary Γ_h and as such, are both equal to zero. Consequently, the finite-difference problem (12.5), (12.6) always has a unique solution for any given right-hand side $f^{(h)}$.

Let us now prove the stability estimate (12.8). According to formula (12.7), for an arbitrary polynomial of the second degree:

$$P(x,y) = ax^2+bxy+cy^2+dx+ey+f,$$

the following equality holds:

$$\boldsymbol{\Lambda}_h[P]_h = \frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 P}{\partial y^2}, \tag{12.12}$$

because the fourth derivatives that appear in the remainder of Taylor's formula (12.7) are equal to zero. (Formula (12.12) is also true for cubic polynomials.)

Let $\varphi_{mn}\equiv\varphi(x_m,y_n)$ and $\psi_{mn}=\psi(s_{mn})$ be the right-hand sides of system (12.5), (12.6), and introduce an auxiliary grid function $P^{(h)}=\{P_{mn}\}$ defined on Ω_h:

$$P_{mn} = \frac{1}{4}[2-(x_m^2+y_n^2)]\cdot\max_{(x_k,x_l)\in\Omega_h^0}|\varphi_{kl}| + \max_{(x_k,x_l)\in\Gamma_h}|\psi_{kl}|.$$

As this function is obviously the trace of a quadratic polynomial on the grid Ω_h, formula (12.12) yields:

$$\boldsymbol{\Lambda}_h P^{(h)}\Big|_{(x_m,y_n)} = -\max_{(x_k,x_l)\in\Omega_h^0}|\varphi_{kl}|, \quad (x_m,y_n)\in\Omega_h^0.$$

Therefore, at the interior grid nodes, i.e., for $(x_m,y_n)\in\Omega_h^0$, the difference between the solution $u^{(h)}$ of problem (12.5), (12.6) and the function $P^{(h)}$ satisfies:

$$\boldsymbol{\Lambda}_h[u^{(h)}-P^{(h)}] = \boldsymbol{\Lambda}_h u^{(h)} - \boldsymbol{\Lambda}_h P^{(h)} = \varphi_{mn} + \max_{(x_k,x_l)\in\Omega_h^0}|\varphi_{kl}| \geq 0.$$

Then, according to Lemma 12.1, the difference $u^{(h)}-P^{(h)}$ assumes its maximum value at the boundary Γ_h. However, at the boundary this difference is non-positive:

$$\begin{aligned} u^{(h)}\Big|_{\Gamma_h} - P^{(h)}\Big|_{\Gamma_h} &= \psi_{mn}-P_{mn} \\ &= [\psi_{mn} - \max_{(x_k,x_l)\in\Omega_h}|\psi_{kl}|] + \frac{1}{4}[x_m^2+y_n^2-2]\cdot\max_{(x_k,x_l)\in\Omega_h^0}|\varphi_{kl}|, \end{aligned}$$