experiments corroborate that when the grid is refined, the solution $u^{(h)}$ of problem (11.17) with piecewise monotone and piecewise smooth initial data $\psi(x)$ converges to some function $u(x,t)$ that has a finite number of discontinuities. In doing so, the convergence is uniform outside of any neighborhood of the shocks.

Of course, the Godunov scheme, for which the fluxes are computed based on the solution to a specially chosen Riemann problem, is not the only conservative scheme for the problem (11.2), (11.10). Conservative schemes can be obtained using various approaches, in particular, the one based on the predictor-corrector idea. For example, the well-known MacCormack scheme is conservative:

$$\frac{\tilde{u}_m - u_m^p}{\tau} + \frac{1}{h}\left[\frac{(u_{m+1}^p)^2}{2} - \frac{(u_m^p)^2}{2}\right] = 0,$$

$$\frac{u_m^{p+1} - (u_m^p + \tilde{u}_m)/2}{\tau/2} + \frac{1}{h}\left[\frac{(\tilde{u}_m)^2}{2} - \frac{(\tilde{u}_{m-1})^2}{2}\right] = 0. \tag{11.21}$$

Proving this property is the subject of Exercise 2. Let us note, however, that even though the MacCormack scheme is consistent and conservative, its solutions do not always converge to the correct generalized solution of problem (11.2). Under certain conditions, the solution obtained by scheme (11.21) may contain a non-physical unstable discontinuity of the type shown in Figure 11.6(b), see [Tho99, Section 9.5].

Yet another example of a conservative predictor-corrector scheme consists of the implicit non-conservative predictor stage:

$$\frac{\tilde{u}_m - u_m^p}{\tau/2} + u_m^p \frac{\tilde{u}_m - \tilde{u}_{m-1}}{h} = 0 \tag{11.22a}$$

followed by the corrector stage rendered via scheme (11.17), where

$$U_{m+1/2}^{p+1/2} = \frac{1}{2}(\tilde{u}_m + \tilde{u}_{m+1}). \tag{11.22b}$$

Scheme (11.22a), (11.22b), (11.17) is second order accurate on smooth solutions of the Burgers equation. Its other properties are to be studied in Exercise 3.

A detailed discussion on numerical solution of conservation laws can be found, e.g., in [Tho99, Chapter 9], as well as in [LeV02].

Exercises

1. Prove the maximum principle (11.20).

2. Prove that the MacCormack scheme (11.21) is conservative.

    Hint. Define $\left(U_{m+1/2}^{p+1/2}\right)^2 = \frac{(u_{m+1}^p)^2}{2} + \frac{(\tilde{u}_m)^2}{2}$.

3. Analysis of scheme (11.22a), (11.22b), (11.17).

    a) Show that scheme (11.22a), (11.22b), (11.17) is conservative.

    b) For the linearized equation with the constant propagation speed: $u_t + au_x = 0$, show that the scheme is stable in the von Neumann sense for any $r = \tau/h$.