[which is not present in the otherwise similar scheme (11.14)]:

$$\begin{gathered}\frac{u_m^{p+1}-u_m^p}{\tau}+u_m^p\frac{u_m^p-u_{m-1}^p}{h}=\mu\frac{u_{m+1}^p-2u_m^p+u_{m-1}^p}{h^2},\\ m=0,\pm1,\pm2,\dots,\quad p=0,1,\dots,\\ u_m^0=\psi(mh).\end{gathered}\tag{11.15}$$

Assume that $h \longrightarrow 0$, and also that a sufficiently small time step $\tau = \tau(h,\mu)$ is chosen so that to ensure stability. Then the solution $u^{(h)} = \{u_m^p\}$ of problem (11.15) converges to the generalized solution of problem (11.2), (11.10). The convergence is uniform in space and takes place everywhere except on an arbitrarily small neighborhoods of the discontinuities of the generalized solution. To ensure the convergence, the viscosity $\mu = \mu(h)$ must vanish as $h \longrightarrow 0$ with a certain (sufficiently slow) rate. Various techniques based on the idea of artificial dissipation (artificial viscosity) have been successfully implemented for the computation of compressible fluid flows, see, e.g., [RM67, Chapters 12 & 13] or [Tho95, Chapter 7]. Their common shortcoming is that they tend to smooth out the shocks. As an alternative, one can explicitly build the desired conservation law into the structure of the scheme used for computing the generalized solutions to problem (11.2).

11.2.2 The Method of Characteristics

In this method, we use special formulae to describe the evolution of discontinuities that appear in the process of computation, i.e., as the time elapses. These formulae are based on the condition (11.10) that must hold at the location of discontinuity. At the same time, in the regions of smoothness we use the differential form of the conservation law, i.e., the Burgers equation itself: $u_t + uu_x = 0$.

The key components of the method of characteristics are the following. For simplicity, consider a uniform spatial grid $x_m = mh$, $m = 0, \pm1, \pm2, \dots$. Suppose that the function $\psi(x)$ from the initial condition $u(x,0) = \psi(x)$ is smooth. From every point $(x_m, 0)$, we will "launch" a characteristic of the differential equation $u_t + uu_x = 0$. In doing so, we will assume that for the given function $\psi(x)$, we can always choose a sufficiently small τ such that on any time interval of duration τ, every characteristic intersects with no more than one neighboring characteristic. Take this τ and draw the horizontal grid lines $t = t_p = p\tau$, $p = 0, 1, 2, \dots$. Consider the intersection points of all the characteristics that emanate from the nodes $(x_m, 0)$ with the straight line $t = \tau$, and transport the respective values of the solution $u(x_m, 0) = \psi(x_m)$ along the characteristics from the time level $t = 0$ to these intersection points.

If no two characteristics intersect on the time interval $0 \le t \le \tau$, then we perform the next step, i.e., extend all the characteristics until the time level $t = 2\tau$ and again transport the values of the solution along the characteristics to the points of their intersection with the straight line $t = 2\tau$. If there are still no intersections between the characteristics for $\tau \le t \le 2\tau$, then we perform yet another step and continue this way until on some interval $t_p \le t \le t_{p+1}$ we find two characteristics that intersect.