From the definition of $w(\lambda)$ it is easy to see that $-u^p$ is the residue of the vector-function $\lambda^{p-1}w(\lambda)$ at infinity:

$$u^p = \frac{1}{2\pi i} \oint\limits_{|\lambda|=\gamma} \lambda^{p-1} w(\lambda) d\lambda = -\frac{1}{2\pi i} \oint\limits_{|\lambda|=\gamma} \lambda^p (\boldsymbol{R} - \lambda \boldsymbol{I})^{-1} u^0 d\lambda.$$

As $u^p = \boldsymbol{R}^p u^0$, the last equality is equivalent to (10.158). ▯

Altogether, we have seen that the question of stability for evolution finite-difference schemes on finite intervals reduces to studying the spectra of the families of the corresponding transition operators $\{\boldsymbol{R}_h\}$. More precisely, we need to find out whether the spectrum for a given family of operators $\{\boldsymbol{R}_h\}$ belongs to the unit disk $|\lambda| \leq 1$. *If it does, then the scheme is either stable or, in the worst case scenario, it may only develop a mild instability.*

Let us now show how we can actually calculate the spectrum of a family of operators. To demonstrate the approach, we will exploit the previously introduced example (10.142a), (10.142b). *It turns out that the algorithm for computing the spectrum of the family of operators $\{\boldsymbol{R}_h\}$ coincides with the Babenko-Gelfand procedure described in Section 10.5.1.* Namely, we need to introduce three auxiliary operators: $\overleftrightarrow{\boldsymbol{R}}$, $\overrightarrow{\boldsymbol{R}}$, and $\overleftarrow{\boldsymbol{R}}$. The operator $\overleftrightarrow{\boldsymbol{R}}$, $v = \overleftrightarrow{\boldsymbol{R}} u$, is defined on the linear space of bounded grid functions $u = \{\dots, u_{-1}, u_0, u_1, \dots\}$ according to the formula:

$$v_m = (1-r)u_m + ru_{m+1}, \quad m = 0, \pm 1, \pm 2, \dots, \tag{10.159}$$

which is obtained from (10.142a), (10.142b) by removing both boundaries. The operator $\overrightarrow{\boldsymbol{R}}$ is defined on the linear space of functions $u = \{u_0, u_1, \dots, u_m, \dots\}$ that vanish at infinity: $|u_m| \longrightarrow 0$ as $m \longrightarrow +\infty$. It is given by the formula:

$$v_m = (1-r)u_m + ru_{m+1}, \quad m = 0, 1, 2, \dots, \tag{10.160}$$

which is obtained from (10.142a), (10.142b) by removing the right boundary. Finally, the operator $\overleftarrow{\boldsymbol{R}}$ is defined on the linear space of functions $\{\dots, u_m, \dots, u_0, \dots, u_{M-1}, u_M\}$ that satisfy: $|u_m| \longrightarrow 0$ as $m \longrightarrow -\infty$. It is given by the formula:

$$\begin{aligned} v_m &= (1-r)u_m + ru_{m+1}, \quad m = \dots, -1, 0, 1, \dots, M-1, \\ v_M &= u_M, \end{aligned} \tag{10.161}$$

which is obtained from (10.142a), (10.142b) by removing the left boundary. Note that the spaces of functions for the operators $\overrightarrow{\boldsymbol{R}}$ and $\overleftarrow{\boldsymbol{R}}$ are defined on semi-infinite grids $m = 0, 1, 2, \dots$ and $m = \dots, -1, 0, 1, \dots, M$, respectively.

None of the operators $\overleftrightarrow{\boldsymbol{R}}$, $\overrightarrow{\boldsymbol{R}}$, or $\overleftarrow{\boldsymbol{R}}$ depend on h. We will show that *the combination of all eigenvalues of these three auxiliary operators yields the spectrum of the family of operators $\{\boldsymbol{R}_h\}$.* In Section 10.5.1, we have, in fact, already computed the eigenvalues of the operators $\overleftrightarrow{\boldsymbol{R}}$ and $\overrightarrow{\boldsymbol{R}}$. For the operator $\overleftrightarrow{\boldsymbol{R}}$, the eigenvalues are all