be a set of interpolation nodes, such that $\alpha \le x_{k-j} < x_{k-j+1} < \ldots < x_{k-j+s} \le \beta$. *Then, to approximately evaluate the derivatives*

$$\frac{d^q f(x)}{dx^q}, \qquad q = 1, 2, \ldots, s,$$

of the function $f(x)$ *on the interval* $x_k \le x \le x_{k+1}$, *one can employ the interpolating polynomial* $P_s(x, f_{kj})$ *and set*

$$\frac{d^q f(x)}{dx^q} \approx \frac{d^q}{dx^q} P_s(x, f_{kj}), \qquad x_k \le x \le x_{k+1}. \tag{2.34}$$

In so doing, the approximation error will satisfy the estimate:

$$\begin{aligned} &\max_{x_k \le x \le x_{k+1}} \left| \frac{d^q f(x)}{dx^q} - \frac{d^q}{dx^q} P_s(x, f_{kj}) \right| \\ &\qquad \le \frac{1}{(s-q+1)!} \max_{x_{k-j} \le x \le x_{k-j+s}} |f^{(s+1)}(x)| (x_{k-j+s} - x_{k-j})^{s-q+1}. \end{aligned} \tag{2.35}$$

PROOF Consider an auxiliary function $\varphi(x) \stackrel{\text{def}}{=} f(x) - P_s(x, f_{kj})$; it obviously vanishes at all $s+1$ interpolation nodes $x_{k-j}, x_{k-j+1}, \ldots, x_{k-j+s}$. Therefore, its first derivative $\varphi'(x)$ will have a minimum of s zeros on the interval $x_{k-j} \le x \le x_{k-j+s}$, because according to the Rolle (mean value) theorem, there is a zero of the function $\varphi'(x)$ in between any two neighboring zeros of $\varphi(x)$. Similarly, the function $\frac{d^q \varphi(x)}{dx^q}$ will have at least $s-q+1$ zeros on the interval $x_{k-j} \le x \le x_{k-j+s}$. This implies that the derivative $\frac{d^q f(x)}{dx^q}$ and the polynomial $\frac{d^q}{dx^q} P_s(x, f_{kj})$ of degree no greater than $s-q$ coincide at $s-q+1$ distinct points. In other words, the polynomial $P_s^{(q)}(x, f_{kj})$ can be interpreted as an interpolating polynomial of degree no greater than $s-q$ for the function $f^{(q)}(x)$ on the interval $x_{k-j} \le x \le x_{k-j+s}$, built on some set of $s-q+1$ interpolation nodes.

Moreover, the function $f^{(q)}(x)$ has a continuous derivative of order $s-q+1$ on $[\alpha, \beta]$:

$$\frac{d^{s-q+1}}{dx^{s-q+1}} f^{(q)}(x) = \frac{d^{s+1}}{dx^{s+1}} f(x).$$

Consequently, one can use Theorem 2.6 and, by setting $\alpha = x_{k-j}$, $\beta = x_{k-j+s}$, obtain the following estimate [cf. formula (2.27)]:

$$\begin{aligned} &\max_{x_{k-j} \le x \le x_{k-j+s}} \left| f^{(q)}(x) - P_s^{(q)}(x, f_{kj}) \right| \\ &\qquad \le \frac{1}{(s-q+1)!} \max_{x_{k-j} \le x \le x_{k-j+s}} |f^{(s+1)}(x)| (x_{k-j+s} - x_{k-j})^{s-q+1}. \end{aligned}$$

As $\alpha \le x_{k-j} \le x_k < x_{k+1} \le x_{k-j+s} \le \beta$, it immediately yields (2.35). ∎